Chapter 1

Order of Magnitude Astrophysics

Oftentimes in astrophysics, since we are dealing with such large scales, it is convenient to do very rough approximations in order to find simply orders of magnitude. Such calculations inevitably give us the correct dependence on physical quantities, and so help us to understand the physics of very large systems.

As an example, we consider a sphere of radius R and mass M in which the only force present is gravity (that is, there is no pressure force competing). We wish to examine the amount of time it takes for the sphere to collapse in on itself to a point. Ignoring general relativity, the equation of motion is given roughly by

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2}.$$

If we assume that the density ρ is constant, then we can write the total mass as

$$M = \frac{4\pi}{3}R^3\rho$$

so that

$$\begin{aligned} \frac{d^2r}{dt^2} &\approx -\frac{4\pi G\rho}{3}r \\ \frac{d^2r}{dt^2} &\approx -\frac{R}{\tau_{ff}^2}. \end{aligned}$$

Setting the above right hand sides equal and dropping the 4/3, which is of order unity, we get

$$\frac{R}{\tau_{ff}^2} \approx G\rho R,$$

and thus the *free fall time* is given by

$$\tau_{ff} \approx \sqrt{\frac{1}{G\rho}}. \tag{1.1}$$

We shall derive the exact expression shortly, but for comparison, it is given by

$$\tau_{ff} = \sqrt{\frac{3\pi}{32G\rho}}. \tag{1.2}$$

Note that as advertised, the above back of the envelope calculation indeed gave us the correct dependence on the physical quantities. Thus, it is often advisable to attempt such a calculation first in case the full blown derivation turns out to be incredibly complicated!

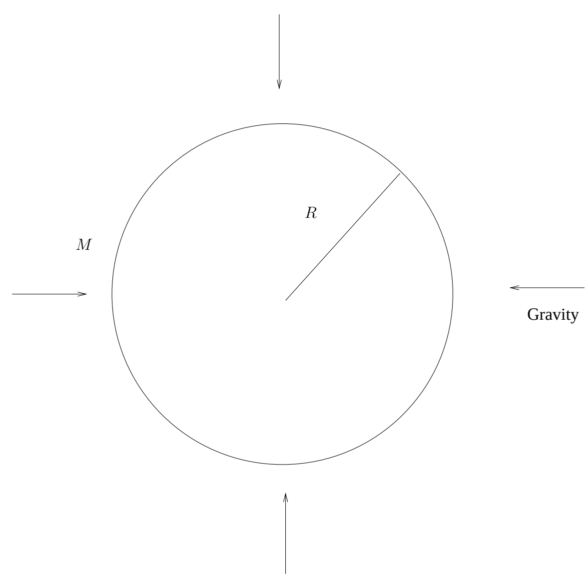


Figure 1.1: A sphere of matter with total mass M and radius R acting only under the influence of gravity. Since there is no pressure counteracting the gravitational force, the time to collapse from $r = R$ to $r = 0$ is called the free fall time.

Chapter 2

Physics of Compact Objects

2.1 Free Fall Time

In Chapter 1, we approximated the free fall time τ_{ff}. Now we perform the exact calculation.

Imagine the "theorist's star." That is, imagine an object which is a perfect sphere, has no rotation, no magnetic fields, and so forth. The average density is

$$\langle \rho \rangle = \frac{M}{(4/3)\pi R^3}. \tag{2.1}$$

To examine the mechanical structure, we use only Newton's second law. First, we write the mass dm contained within the shell of inner radius r and outer radius $r + dr$ as

$$dm = 4\pi r^2 \rho dr$$

so that the mass contained within a sphere of radius r is

$$m(r) = \int_0^R 4\pi r^2 \rho(r) dr. \tag{2.2}$$

Then Newton's second law tells us that

$$dmg = dm\frac{d^2r}{dt^2}.$$

The forces due to gravity and pressure are

$$\begin{aligned}
F_G &= -dmg = -dm\frac{Gm}{r^2} \\
F_P &= 4\pi r^2 P(r) - 4\pi r^2 \left[P(r) + \frac{dP}{dr}dr\right] \\
&= -\frac{1}{\rho}4\pi r^2 \rho \frac{dP}{dr}dr \\
&= -\frac{1}{\rho}\frac{dP}{dr}dm
\end{aligned}$$

Putting it all together, we get the equation of motion

$$\frac{d^2r}{dt^2} = -g - \frac{1}{\rho}\frac{dP}{dr}. \tag{2.3}$$

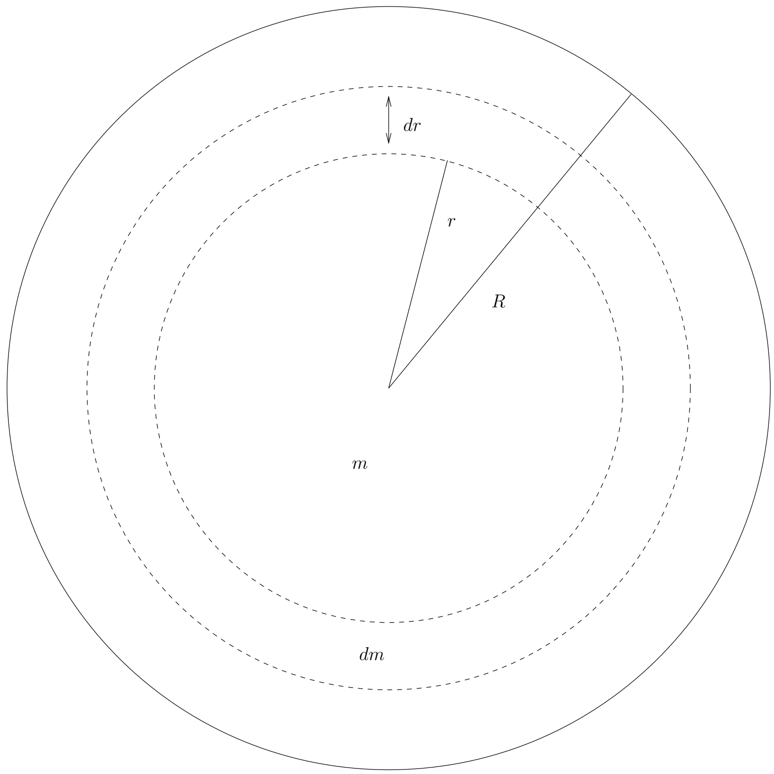


Figure 2.1: The theorist's star.

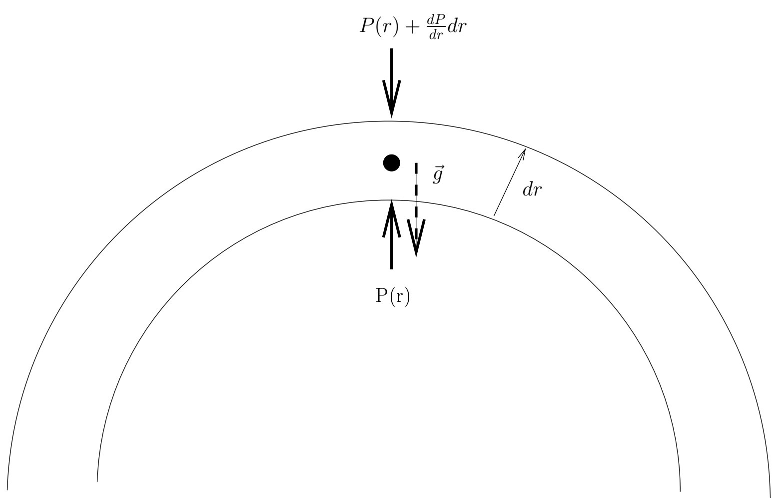


Figure 2.2: A blown up picture of the shell. The forces acting on the mass dm are also shown.

Sun	1 hour
White Dwarf	1 s
Neutron Star	0.1 ms

Table 2.1: Some approximate free fall times. Given these values, one can easily imagine how different the universe would be if G were just a few orders of magnitude different!

To get the free fall time (also known as the *dynamical time*), imagine that there are no pressure forces. Then the equation of motion becomes simply

$$\frac{d^2r}{dt^2} = -\frac{Gm(r)}{r^2}. \tag{2.4}$$

Making some approximations,

$$\frac{d^2r}{dt^2} \approx -\frac{R}{\tau^2} \tag{2.5}$$

and

$$-\frac{Gm(r)}{r^2} \approx -\frac{GM}{R^2} \tag{2.6}$$

Combining these according to Equation 2.4,

$$\frac{R}{\tau^2} \approx \frac{GM}{R^3} \sim G\rho$$

so

$$\tau_{ff} = \frac{1}{\sqrt{G\rho}}. \tag{2.7}$$

This is exactly what we saw earlier (Equation 1.1). Some examples of approximate free fall times are given in Table 2.1.

2.2 Hydrostatic Equilibrium

Since we know that stars do not collapse under their own mass (nor do they explode very quickly due to pressure forces), then they must be in *hydrostatic equilibrium*. That is,

$$\frac{dP}{dr} = -g\rho \tag{2.8}$$

2.2.1 Central Pressure

To estimate the central pressure P_C of a star, we set

$$\frac{dP}{dr} \approx \frac{P_0 - P_C}{R},$$

where $P_0 \approx 0$ is the pressure at the surface. Then

$$\frac{dP}{dr} \approx -\frac{P_C}{R} = -\frac{Gm(r)}{r^2}\rho,$$

thus

$$P_C = \frac{GM^2}{R^4}. \tag{2.9}$$

Thus, the central pressure of the Sun is on the order[1] 10^{16} dyne cm^{-2} (*cf.* atmospheric pressure at 10^6 dyne cm^{-2}).

2.2.2 Another Look at Hydrostatic Equilibrium

We can also describe the strength of gravity using the gravitational potential $E_{\rm pot}$:

$$E_{\rm pot} \approx \int_0^R \frac{Gm(R)}{r} dm. \tag{2.10}$$

We can then calculate the average pressure needed to balance the force of gravity. Average pressure is

$$\langle P \rangle = \frac{\int_0^R 4\pi r^2 P(r) dr}{\int_0^R 4\pi r^2 dr} = \frac{1}{V} \int_0^R r^2 P(r) dr,$$

where V is the volume of the sphere. Intregrating by parts,

$$\int_0^R 4\pi r^2 P(r) dr = \frac{4\pi}{3} \left[P(r) r^3 \right]_0^R - \frac{1}{3} \int_0^R 4\pi r^3 \frac{dP}{dr} dr,$$

and the first term on the right hand side is zero. Using Equation 2.8,

$$\begin{aligned} 4\pi r^3 \frac{dP}{dr} dr &= -4\pi r^3 \rho \frac{Gm}{r^2} dr \\ &= -\frac{Gm}{r} dm. \end{aligned}$$

Thus,

$$\langle P \rangle = -\frac{1}{3} \frac{E_{\rm pot}}{V}. \tag{2.11}$$

In general, a pressure is a force per area:

$$P = \frac{\Delta F}{\Delta A} = \frac{\Delta p}{\Delta A \Delta t},$$

so sometimes pressure is called "momentum flux."

Consider particles in the box shown in Figure 2.3. The change in volume ΔV is

$$\Delta V = v_x \Delta t \Delta A$$

while the change in the number of particles is

$$\Delta N = \frac{1}{2} n v_x \Delta t \Delta A.$$

Then the change in momentum is

$$\Delta p = \Delta N 2 p_x = n \Delta A p_x v_x \Delta t.$$

Given that $\langle \vec{p} \cdot \vec{v} \rangle = \langle p_x v_x \rangle + \langle p_y v_y \rangle + \langle p_y v_y \rangle$, and using the isotropic pressure condition (i.e., there is no preferred direction),

[1] 1 N = 10^5 dyne and 1 J = of 10^7 erg.

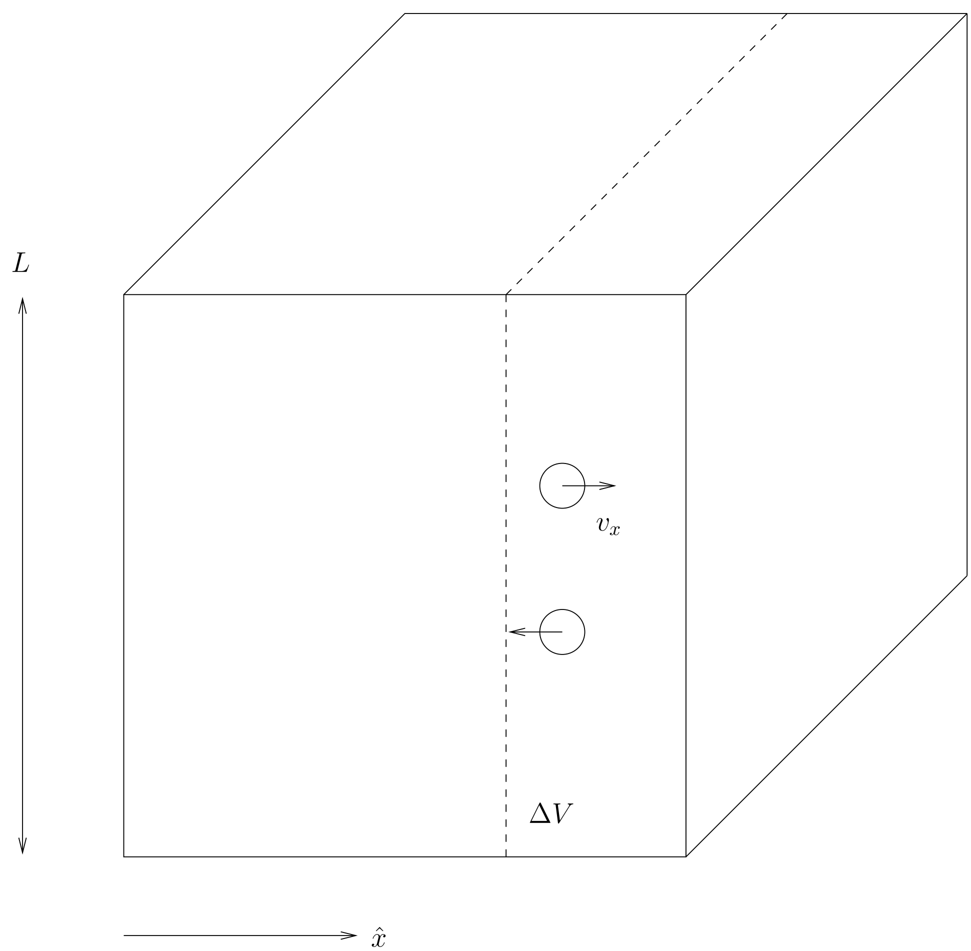


Figure 2.3: Particles contained within a volume $V = L^3$ and density $N = nV$. There are as many particles with velocity v_x as those with velocity $-v_x$.

$$\langle \vec{p} \cdot \vec{v} \rangle = 3 \langle p_x v_x \rangle$$

so

$$P = \frac{\Delta p}{\Delta A \Delta t} = \frac{n}{3} \langle \vec{p} \cdot \vec{v} \rangle . \tag{2.12}$$

In the case of a nonrelativistic gas, we have that $p = m_0 v$ and $T = (1/2) m_0 v^2$ (so $pv = 2T$). Then

$$\begin{aligned} P &= \frac{1}{3} \frac{N}{V} 2T \\ &= \frac{2}{3} \frac{NT}{V} \\ P &= \frac{2}{3} u_T, \end{aligned} \tag{2.13}$$

where u_T is the kinetic energy density.

We compare this to the ultrarelativistic case in which $v = c$. Then $T = mc^2 = pc$ (so $pv = T$). Then

$$\begin{aligned} P &= \frac{1}{3} \frac{NT}{V} \\ P &= \frac{1}{3} u_T. \end{aligned} \tag{2.14}$$

In one case, there's a 2/3, while in the other there's a 1/3!

2.3 Total Energy and Stability

2.3.1 Nonrelativistic Gas

The equilibrium condition demands that we have

$$\langle P \rangle = -\frac{1}{3} \frac{E_{\text{pot}}}{V},$$

and from the random motion of particles, we have

$$\langle P \rangle = \frac{2}{3} \frac{T}{V}.$$

Thus, for the system to be in hydrostatic equilibrium, we set these two expressions equal. Doing so, and we find

$$2T = -E_{\text{pot}}. \tag{2.15}$$

This is known as the *virial theorem* and is one of the most useful formulae in all of astrophysics. Despite being simply a reformulation of the hydrostatic equilibrium condition, the virial theorem applies to *any* bound system, no matter the particle type.

The total energy is simply

$$E = T + E_{\text{pot}}.$$

If $E < 0$ the system is bound.

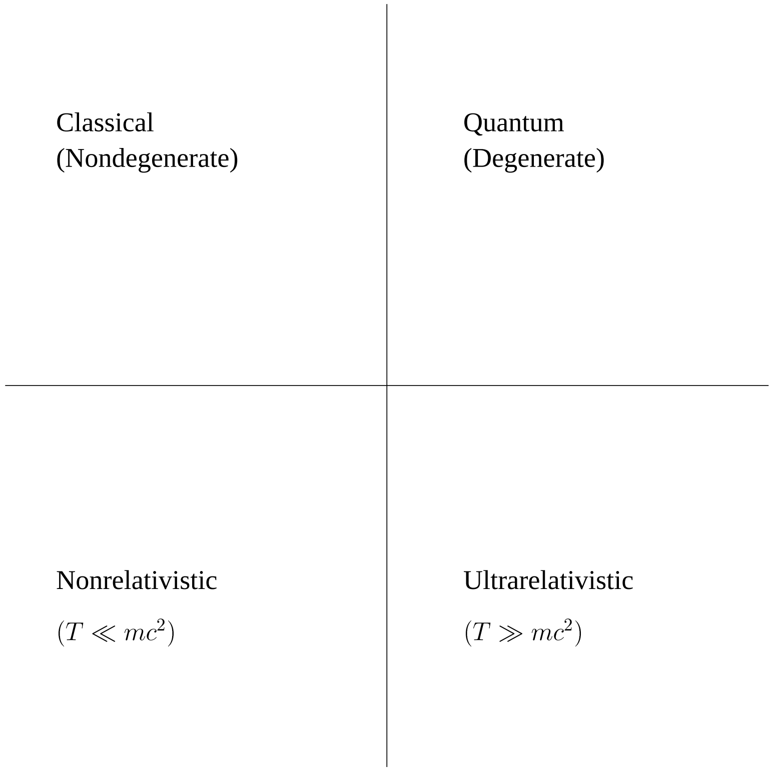


Figure 2.4: The four pressure regimes.

2.3.2 Ultrarelativistic Gas

Just as in the nonrelativistic case, here we need to have the average pressure to be

$$\langle P \rangle = -\frac{1}{3}\frac{E_{\text{pot}}}{V},$$

but we have

$$\langle P \rangle = \frac{1}{3}\frac{T}{V}.$$

So now $T = -E_{\text{pot}}$, and so

$$E = 0 \tag{2.16}$$

in the relativistic case. This means that at best, an ultrarelativistic gas can be only marginally bound (i.e., it is unstable).

2.4 Pressure Inside Stars

From thermodynamics, we know that the equation of state to determine a pressure is a function of density n and temperature T, as well as perhaps some other variables: $P = P(n, T, \ldots)$. There are four separate regimes of pressure, as shown in Figure 2.4.

2.4.1 Classical Nonrelativistic Gas

In the classical and nonrelativistic regime,

$$P = \frac{2}{3} u_T,$$

where, recall,

$$u_T = n \langle T \rangle$$

(here $\langle T \rangle$ represents the average kinetic energy per particle). Because this is classical, we can treat the gas as ideal, and so

$$\langle T \rangle = \frac{3}{2} k_B T. \tag{2.17}$$

Then

$$u_T = \frac{3}{2} n k_B T,$$

and so we have (not surprisingly), the ideal gas law as our equation of state:

$$P = n k_B T. \tag{2.18}$$

2.4.2 Quantum Nonrelativistic Gas

Next we examine the boundary between the classical and quantum regimes, as illustrated schematically in Figure 2.5. Essentially, quantum effects become important when the average distance l between particles becomes comparable to the de Broglie wavelength. That is,

$$n = \frac{N}{V} \sim \frac{1}{l^3},$$

so that $l = n^{-1/3}$. Recalling that the de Broglie wavelenth is $\lambda = h/p$, we need to be wary of quantum mechanics when

$p < hn^{1/3}$. This is supposed to uselesssim, but for some reason it's not working... (2.19)

This tells us that slow moving, dense particles are in danger of becoming degenerate. We can also relate this to the temperature since we know both the ideal gas expression for kinetic energy (Equation 2.17) and the usual expression for kinetic energy ($T = (1/2)mv^2 = p^2/2m$). We find that

$$k_B T \sim \frac{p^2}{m}. \tag{2.20}$$

Hence quantum mechanics is important when the temperature is proportional to $n^{2/3}$.

In the quantum regime, we consider the 6-D phase space consisting of (x, y, z, p_x, p_y, p_z). In phase space, there exist quantum cells of volume dVd^3p, where the volume element dV is related to the position part of phase space and is approximately[2] h^3. Being in the quantum degenerate limit, the Pauli exclusion principle[3] must also be considered.

Consider a fermionic gas which is packed as tightly as possible (i.e., it is completely degenerate). Then according to Pauli, there will be a greatest momentum, known as the *fermi momentum*, p_F (see Figure 2.7). To pack the fermions as tightly as possible, we can fit

[2]This comes from the Heisenberg uncertainty principle: $dxdp_x \geq h/2$.

[3]Here, the statement of it that we will use is "No more than two fermions may be in one quantum cell at any one time."

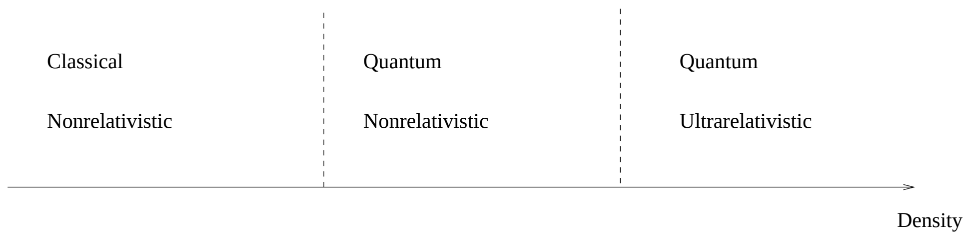


Figure 2.5: An illustration of the boundaries between several regimes. As density increases, so does the energy, and thus the system must be treated differently at different densities.

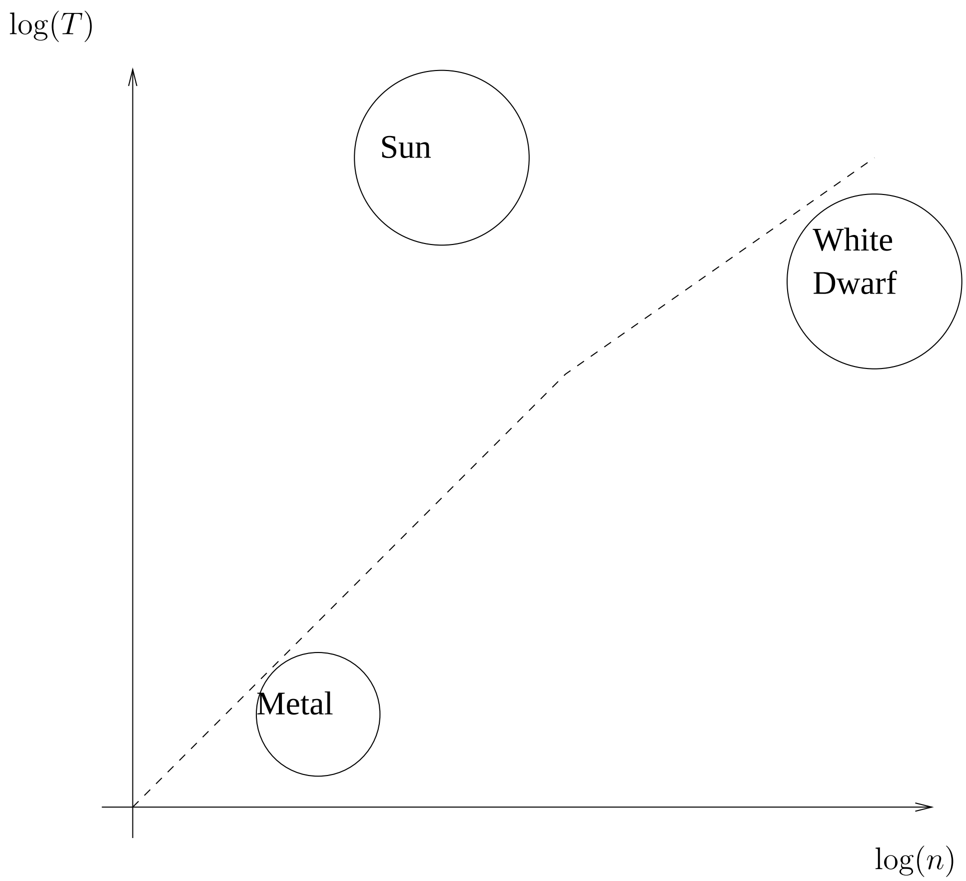


Figure 2.6: A schematic of what pressure regime certain objects are most commonly in.

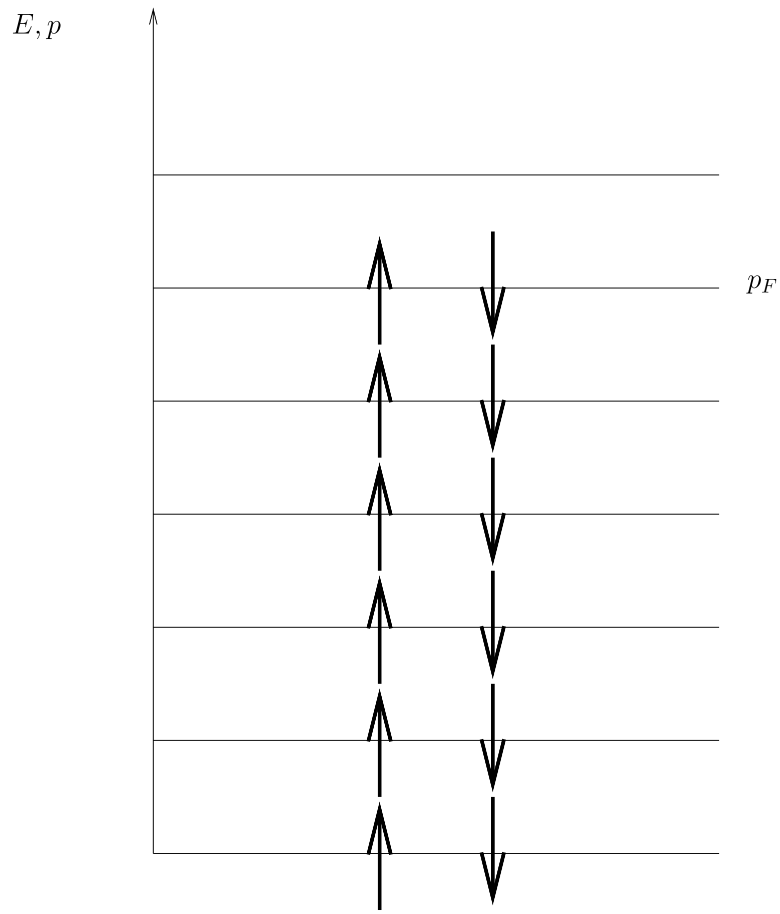


Figure 2.7: A schematic of a completely degenerate fermion gas. The Fermi momentum is the highest momentum and is sometimes referred to as the "surface of the fermi sea."

$$N = 2\frac{\frac{4\pi}{3}p_F^2 V}{h^3}$$

particles. Note that $4\pi/3p_F^3$ is the volume of a sphere of radius p_F in momentum space. The factor of 2 out front is due to a spin degeneracy of two. We often write the above number instead as a number density:

$$n = \frac{8\pi}{3h^3}p_F^3 \tag{2.21}$$

This then gives us the Fermi momentum p_F:

$$p_F = h\left(\frac{3}{8\pi}\right)^{1/3} n^{1/3}. \tag{2.22}$$

Recall that the pressure for a nonrelativistic gas is given by $P = (2/3)u_{\text{kin}}$, where $u_{\text{kin}} = n\langle T\rangle$. We approximate the number density n as

$$n \approx \frac{p_F^3}{h^3}.$$

Then, since we are still in the nonrelativistic regime, the average kinetic energy is given by

$$\langle T\rangle \approx \frac{p_F^2}{h^3} \sim \frac{p_F^2}{m}.$$

Thus, the kinetic energy density for quantum, nonrelativistic particles is

$$u_{\text{kin}} \approx \frac{1}{h^3 m}p_F^5.$$

A more precise calculation would yield

$$u_{\text{kin}} = \frac{8\pi}{10}\frac{p_F^5}{h^3 m}. \tag{2.23}$$

Combining it all, we get the pressure:

$$P = K_{NR}n^{5/3}, \tag{2.24}$$

where the constant K_{NR} is

$$K_{NR} \equiv \frac{h^2}{5m}\left(\frac{3}{8\pi}\right)^{2/3} \tag{2.25}$$

The $n^{5/3}$ factor effectively "resists" compression. Note also that in the quantum regime, the equation of state no longer depends on termperature. This will be important when we discuss white dwarfs.

2.4.3 Quantum Ultrarelativistic Gas

In the case of a quantum, ultrarelativistic gas, the pressure becomes $P = (1/3)u_{\text{kin}}$. Now,

$$n \approx \frac{p_F^3}{h},$$

and

$$\langle T\rangle \approx p_F c.$$

Then we find that the average kinetic energy density is given approximately by

$$u_{\rm kin} \approx \frac{c}{h^3} p_F^4.$$

It can be shown by a more careful calculation that the exact result is

$$u_{\rm kin} = \frac{2\pi c}{h^3} p_F^4. \tag{2.26}$$

Putting all this together, we get the equation of state for an ultrarelativistic quantum gas to be

$$P = K_{UR} n^{4/3}, \tag{2.27}$$

where the constant K_{UR} is

$$K_{UR} \equiv \frac{hc}{4} \left(\frac{3}{8\pi} \right)^{1/3} \tag{2.28}$$

Note the power of 4/3, compared to 5/3 in the nonrelativistic case. That is, an ultrarelativistic quantum gas resists compression slightly less than a nonrealtivistic quantum gas. Also note that K_{NR} contains h to include quantum mechanics and that K_{UR} adds c to also include relativity.

Finally, note that what mass we talk about depends on the particular case. For example, in white dwarfs, we consider the electrons (so $m = m_e$), and in the case of neutron stars, we discuss the neutrons ($m = m_n$).

2.4.4 The Quantum Nonrelativistic/Ultrarelativistic Boundary

In a classical world, the Fermi momentum $p_F = mv \approx m_p c$. But we also have $p_F \approx hn^{1/3}$ (see Equation 2.19). If we combine these, we find that the boundary density between nonrelativistic and ultrarelativistic effects is

$$n \approx \left(\frac{cm}{h} \right)^3. \tag{2.29}$$

Because of this m^3 dependence, lighter particles become ultrarelativistic much sooner than heavier particles. For example, for an electron gas, the required number density is

$$n = 10^{29} \text{cm}^{-3}.$$

In terms of mass density, this is

$$\rho = \frac{n}{V} \approx m_H n \sim 10^5 \text{g cm}^{-3}.$$

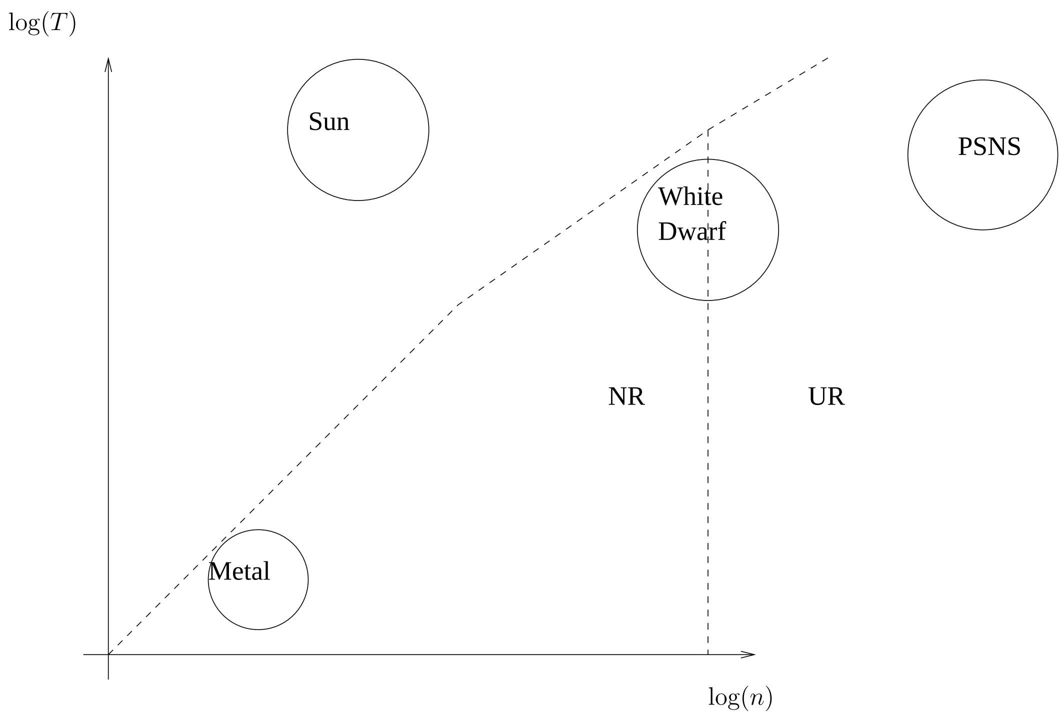


Figure 2.8: Pressure regimes with the boundary between nonrelativistic and ultrarelativistic gases shown. Note that white dwarfs lie on the boundary. Pre-supernova stars (PSNSs) are very unstable due to being entirely in the ultrarelativistic regime.

Chapter 3

Stellar Evolution

3.1 Luminosity

The *luminosity* of a star is

$$L = \frac{\Delta E}{\Delta t}. \tag{3.1}$$

The flux is $\Delta E/(\Delta T \Delta A)$. Stars are approximately blackbodies, so we can utilize the Stefan-Boltzmann law:

$$\frac{\Delta E}{\Delta T \Delta A} = \sigma_{SB} T_{eff}^4. \tag{3.2}$$

The luminosity can then be written

$$L = \frac{\Delta E}{\Delta t} = 4\pi R^2 \sigma_{SB} T_{eff}^4. \tag{3.3}$$

3.2 Hertzsprung-Russel Diagram

The Hertzsprun-Russel (HR) diagram demonstrates that a vast majority of stars spend about 90% of their lifetime in the *main sequence*. Mass increases going diagonally up and to the left. It can be shown that the luminosity is proportional to M^3 and

$$\text{lifetime} \propto \frac{Mc^2}{L} \propto \frac{M}{M^3} \propto M^{-2}.$$

The life cycle of a low mass star involves

Main Sequence → Red Giant → Planetary Nebula → White Dwarf

whereas the life cycle of a high mass star involves

Main Sequence → Red Supergiant → Supernova → Neutron Star or Black Hole

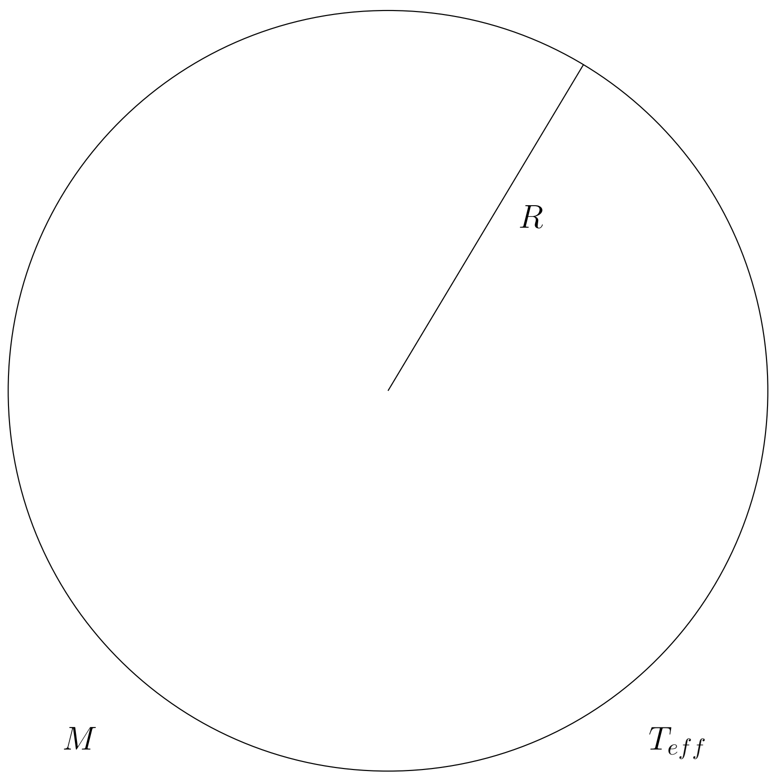


Figure 3.1: Some of the more important characteristics of a star. T_{eff} is the effective surface temperature.

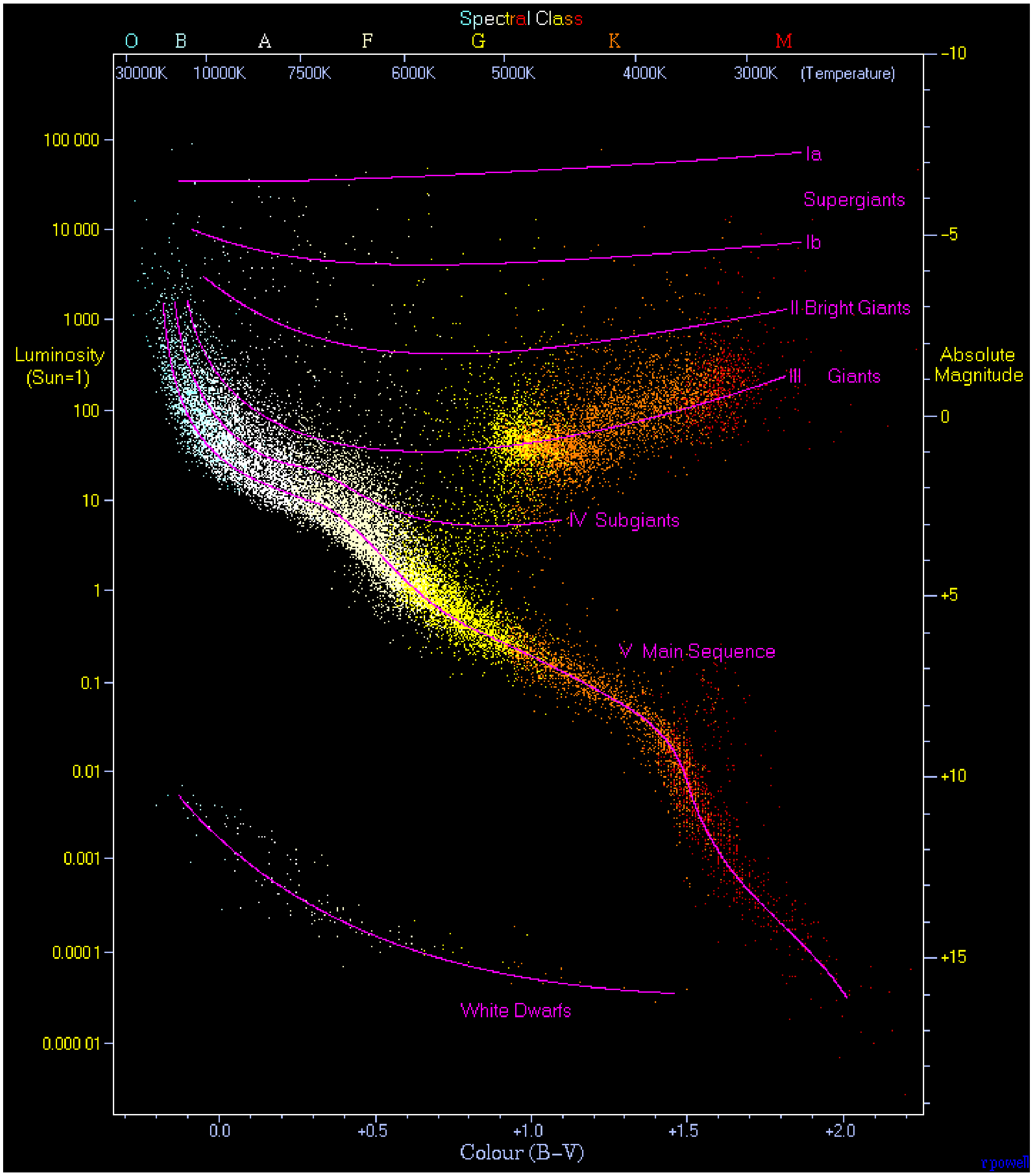


Figure 3.2: The Hertzsprung-Russel diagram. Courtesy Wikipedia.

Chapter 4

White Dwarfs

White dwarfs were first discovered circa 1850 via observation of the Sirius double star system. Sirius A is the brightest star in the night sky, but Sirius B, it's small partner, is a white dwarf with mass approximately $M \sim 1M_\odot$ and radius $R \sim R_\odot/100 \sim R_E$ (these numbers are very typical of a white dwarf). Using these numbers, the average density is found to be

$$\langle\rho\rangle = 10^6 \text{ g cm}^{-3}, \tag{4.1}$$

compared to the average density of the sun of about 1 g cm^{-3}; quite preposterous before quantum mechanics was discovered! Without being able to defer to quantum mechanics, there did not seem to be any way that a white dwarf could be able to support itself under its own gravity.

4.1 Estimate for the Strength of General Relativistic Effects

The important concept is the *Schwarzschild radius*[1]:

$$R_S = \frac{2GM}{c^2}. \tag{4.2}$$

Thus, the solar Schwarzschild radius would be about 3 km.

A sloppy way of deriving this considers Newtonian gravitation. We have that

$$\frac{1}{2}v_{esc}^2 = \frac{GM}{R}.$$

As $v_{esc} \to c$, we get the Schwarzschild radius above. While this "derivation" is completely bogus, it does serve as a useful mnemonic in remembering the formula.

The formal meaning of the Schwarzschild radius is that it gives the circumference of a black hole's event horizon: $2\pi R_S$. Less formally, we can estimate general relativistic effects by comparing R_S to a star's radius. So, for the sun,

$$\frac{R_S}{R_\odot} \approx \frac{3 \text{ km}}{7 \times 10^5 \text{ km}} \sim 10^{-6}.$$

Thus, we can in practice completely ignore general relativity as far as the sun is concerned. For a white dwarf,

$$\frac{R_S}{R} \approx \frac{3 \text{ km}}{6000 \text{ km}} \sim 10^{-4}.$$

[1] We will, of course, reexamine this in more detail later.

Now compare these to neutron stars:

$$\frac{R_S}{R} \approx \frac{3\ \text{km}}{10\ \text{km}} \sim 0.5.$$

Clearly, general relativity is now quite important!

4.2 Basic Properties of White Dwarfs

Recall that for a star to be in hydrostatic equilibrium, we need

$$\langle \rho \rangle = -\frac{1}{3}\frac{E_{\text{pot}}}{V}.$$

For a "not too massive" white dwarf, gravity can be balanced by electron degeneracy pressure. That is,

$$\langle P \rangle \approx P_{NR} = K_{NR} n^{5/3}.$$

In this case, n refers to the number density of the electrons. In order to get this in terms of mass density, we first note that $m_e \sim m_p/2000$, meaning that most of the mass is provided by the nucleons. A white dwarf is composed of mostly carbon, along with some oxygen and helium. For all three of these elements, the ratio of nucleons to electrons is 2:1, thus

$$\rho = 2m_H n \tag{4.3}$$

Then we have that

$$\langle P \rangle \approx \frac{K_{NR}}{(2m_H)^{5/3}} \rho^{5/3}$$

and

$$\begin{aligned} E_{\text{pot}} &\approx -\frac{GM^2}{R}, \\ V &\approx \frac{4\pi}{3} R^3 \sim R^3. \end{aligned}$$

Hence,

$$\frac{K_{NR}}{(2m_H)^{5/3}} \rho^{5/3} \approx -\frac{1}{3}\frac{GM^2}{R^4}.$$

Using $\rho \sim M/R^3$, we can now connect the mass and radius of a white dwarf:

$$\begin{aligned} \frac{M^{5/3}}{R} &\propto \frac{M^2}{R^4} \\ \rho &\propto M^2 \end{aligned} \tag{4.4}$$

So we also have a critical mass.

For $M < M_{\text{crit}}$,

$$\frac{M}{R^3} \propto M^2,$$

so

$$R \propto M^{-1/3}. \tag{4.5}$$

This is known as the *mass-radius relation*.

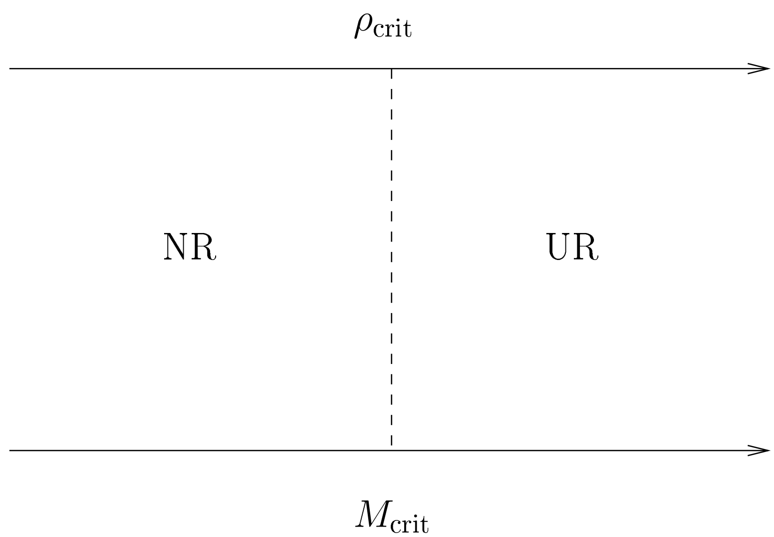


Figure 4.1: The boundary between the nonrelativistic and ultrarelativistic regimes for the white dwarf.

4.3 The Chandrasekhar Limit

As $M \to M_{\text{crit}}$,

$$P \to P_{UR} = \frac{K_{UR}}{(2m_H)^{4/3}} \rho^{4/3}$$

(recall that $K_{UR} \sim hc/8$). In order to balance gravity,

$$P_{UR} > -\frac{1}{3}\frac{E_{pot}}{3} = \frac{1}{3}\frac{GM^2}{R^4}. \tag{4.6}$$

If we again approximate $\rho \sim M/R^3$, we find that

$$M < \left(\frac{3}{8}\right)^{3/2} \left(\frac{hc}{G}\right)^{3/2} \frac{1}{(2m_H)^2} = M_{Ch}. \tag{4.7}$$

This is the famous Chandrasekhar limit:

$$M_{Ch} \approx \frac{1}{4}\left(\frac{3}{8}\right)^{3/2} \left(\frac{hc}{G}\right)^{3/2} \frac{1}{m_H^2} \approx 1.7 M_\odot.$$

A high precision calculation yields the result

$$M_{Ch} = 1.4 M_\odot. \tag{4.8}$$

No white dwarf can exceed this limit (*cf.* the lower bound on mass for a supernova to occur at $8M_\odot$). In order to get an idea of the fundamental nature of this limit, we can rephrase it in terms of other quantities.

4.3.1 The Planck Mass

We define the Planck mass to be

$$m_{Pl} \equiv \left(\frac{hc}{G}\right)^{1/2} \sim 5 \times 10^{-5}\text{g} \sim 10^{19}\,\frac{\text{GeV}}{c^2}. \tag{4.9}$$

This is the lowest mass for which general relativity remains valid.

Recalling the Schwarzschild radius (Equation 4.2), in order for general relativity to be the ruling regime, we must have

$$R_S \geq \lambda_C,$$

where λ_C is the Compton wavelength. To get this fundamental unit of distance, we consider a particle of mass m. By Einstein, the energy of the particle in its rest frame is $E = mc^2$. Also due to Einstein, the energy in terms of the frequency (and hence wavelength) of the particle is $E = h\nu = hc/\lambda$. Setting these expressions for energy equal, we get

$$\lambda_C = \frac{h}{mc}. \tag{4.10}$$

We then have (approximately)

$$\frac{GM}{c^2} \geq \frac{h}{Mc},$$

so in order for general relativity to still be valid, we need

$$M \geq \sqrt{\frac{hc}{G}} \equiv m_{Pl}.$$

So at the Planck scale, general relativity breaks down, and we need a quantum theory of gravity (which, of course, we do not have). Writing the Chandrasekhar limit in terms of the Planck mass,

$$M_{Ch} \approx \frac{1}{4}\left(\frac{3}{8}\right)^{3/2} \frac{m_{Pl}^3}{m_H^2} \approx \frac{m_{Pl}^3}{m_H^2}. \tag{4.11}$$

4.3.2 The Fundamental Strength of Gravity

To give a measure of the strength of the electromagnetic interaction, consider two protons a distance r apart. The electromagnetic (electrostatic) potential energy in this situation is

$$U_{EM} = \frac{e^2}{r}.$$

If we let $r \to \lambda_C$, which in terms of $\hbar$ is

$$\lambda_C = \frac{\hbar}{2\pi mc} \approx \frac{\hbar}{mc},$$

then we find[2] the finestructure constant to be

$$\begin{aligned} \alpha &\approx \frac{U_{EM}(r = \lambda_C)}{m_H c^c} \\ \alpha &= \frac{e^2}{\hbar c} \approx \frac{1}{137}. \end{aligned} \tag{4.12}$$

This is a good relative measure for the strength of the electromagnetic interaction.

We can follow a similar procedure to get the gravitational equivalent. Doing so, we find

$$\alpha_G = \frac{Gm_H^2}{\hbar c} \sim 10^{-38}. \tag{4.13}$$

[2]This is a completely bogus way to "derive" the finestructure constant. However, it does provide a useful mnemonic for remembering it.

Using this to rewrite the Chandresekhar limit again, we find

$$M_{Ch} \approx \left(\frac{hc}{G}\right)^{3/2} \frac{1}{m_H^2} \sim \left(\frac{hc}{Gm_H^2}\right)^{3/2} m_H = \alpha_G^{-3/2} m_H. \tag{4.14}$$

Plugging in numbers, we find that $M_{Ch} \sim 10^{57} m_H \sim 1 M_\odot$. 10^{57} just so happens to be about the approximate number of particles in the sun and nearly any other stellar object.

Interestingly, expressing M_{Ch} in this way demonstrates that if gravity were stronger (i.e., G were larger), then stars would be much *less* massive! Also, since graivty is so weak, something like 10^{57} particles are needed for gravity to overwhelm electromagnetic forces, and hence $M_{Ch} \sim M_\odot$ is a tpical mass for a star.

4.4 Composition and Internal Structure of White Dwarfs

Most white dwarfs are composed of primarily carbon nuclei (in particular, ${}^{12}_{6}\mathrm{C}$, with some helium and oxygen nuclei, and free electrons. Yet the free electrons are degenerate, whereas the ions are not (i.e., they remain ideal). This is because we have in the quantum limit that the de Broglie wavelength is

$$\lambda \approx \frac{h}{\sqrt{m k_B T}},$$

so $\lambda_e \gg \lambda_{\text{ions}}$, meaning that the electrons are degenerate much sooner than the ions.

For the same temperature T throughout a star, we find

$$\lambda(C) \sim \frac{\lambda e^-}{100}. \tag{4.15}$$

In other words, the electrons are "100 times more degenerate" than the carbon ions.

Consider now the total pressure

$$P = P_{e^-} + P_{\text{ion}}.$$

The partial pressure of the electron gas is

$$P_{e^-} = \frac{K_{NR}}{(2m_H)^{5/3}} \rho^{5/3} \approx 10^{22} \text{ dyne cm}^{-2};$$

and that of the ions is

$$P_{\text{ion}} = n k_B T \approx 10^{20} \text{ dyne cm}^{-2}.$$

Thus the greatest contribution to pressure comes from the degenerate electron gas, whereas most of the mass density comes from the heavy ions. Additionally, the temperature really only comes from the ions, since the concept of temperature is rather vague for degenerate gases which do not respond significantly to changes in temperature. The still ideal ions have a temperature on the order of 10^8 K.

4.5 Cooling

To understand the process of cooling in white dwars, we begin with the Stefan-Boltzmann law, Equation 3.2. Now recall the mass-radius relation, here written out more precisely:

$$R = 0.01 R_\odot \left(\frac{M}{M_\odot}\right)^{-1/3}. \tag{4.16}$$

From observation, we find that the effective surface temperature of a white dwarf is $T_{\text{eff}} = 20000$ K (*cf.* the surface temperature of the sun of 6000 K), thus

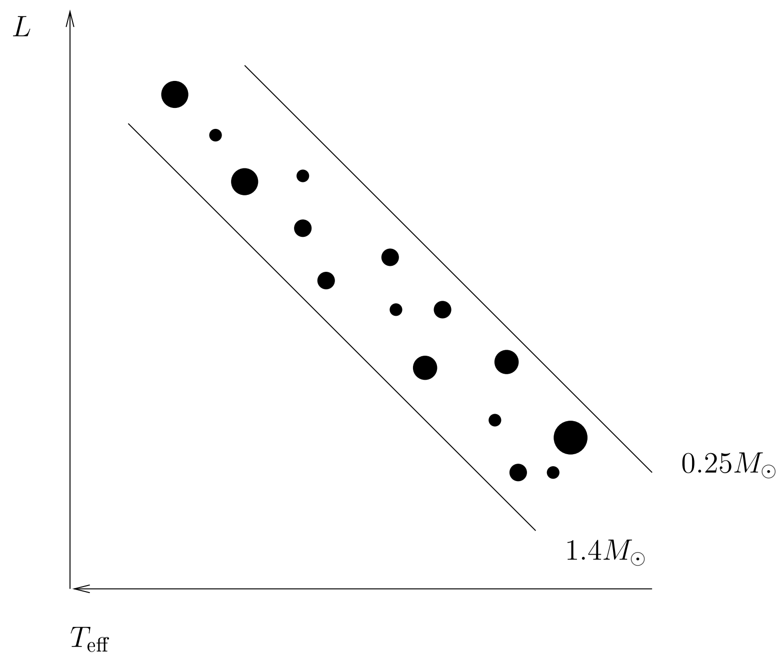


Figure 4.2: All observed white dwarfs lie in between the lines of 1.4 $M_\odot$ and 0.25 $M_\odot$.

$$L \approx 10^{-2} L_\odot \left(\frac{M}{M_\odot}\right)^{-2/3} \left(\frac{T_{\rm eff}}{20,000\ \mathrm{K}}\right)^4. \tag{4.17}$$

So while a white dwarf is much hotter than the sun, it is much less luminous. On a Hertzprung-Russell diagram, all observed white dwarfs lie between two lines as shown in Figure 4.2; this effectively serves as experimental proof of the theoretical model of the white dwarf.

Of course, the mass of a white dwarf doesn't change, so because it still radiates, the ions must cool. In order to get an idea of how long it takes for a white dwarf to radiate away all its thermal energy, we first look at the thermal energy of the ions:

$$E_{Th} \approx \frac{\frac{3}{2}k_B T}{12 m_H} M \sim \frac{\frac{3}{2}k_B T}{12 m_H} M_\odot \sim 10^{48}\ \mathrm{erg}. \tag{4.18}$$

Then we define the cooling time $\tau_{\rm cool}$ as

$$\tau_{\rm cool} \equiv \frac{E_{Th}}{L} \sim 10^9\ \text{years}. \tag{4.19}$$

If we compare this to the age of the universe (the so-called *Hubble time*),

$$\tau_H \approx 13.7 \times 10^9\ \text{years}, \tag{4.20}$$

we immediately see that it is unlikely that *any* white dwarf has ever completely radiated away its thermal energy (such an object is called a *black dwarf*).

Chapter 5

General Relativity

5.1 Newtonian Gravitation

Consider two particles as illustrated in Figure 5.1.

The force on m is given by

$$\vec{F} = \frac{-GMm}{r^2}\hat{r};$$

or we can use the acceleration due to gravity,

$$\vec{g} = \frac{\vec{F}}{m}.$$

So it is already clear that Newton's theory of gravity is all about vectors (*cf.* general relativity, which is all about tensors). It becomes beneficial to introduce the *gravitational potential*

$$\phi = -\frac{GM}{r}, \tag{5.1}$$

such that

$$\vec{g} = -\nabla\phi. \tag{5.2}$$

Rephrasing this,

$$\nabla \cdot \vec{g} = -\nabla^2\phi.$$

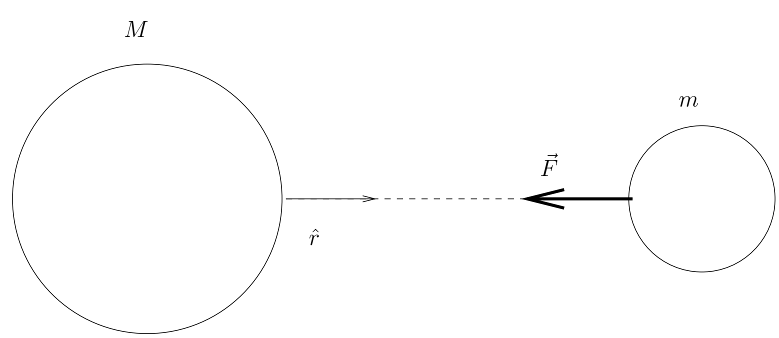


Figure 5.1: Two masses attracting each other due to gravity.

Now consider a mass with spherical symmetry. Then by Gauss's theorem,

$$\oint_V \nabla \cdot \vec{g} dV = \oint \vec{g} \cdot d\vec{A}.$$

But because, $\vec{g}$ is constant on the surface,

$$\oint \vec{g} \cdot d\vec{A} = -g4\pi r^2.$$

Here, $g = Gm/r^2$ where

$$m = \int \rho(\vec{r}) dV.$$

Then

$$\oint_V \nabla \cdot \vec{g} dV = -4\pi G \int \rho dV.$$

In order for this to be true, we must have

$$\nabla \cdot \vec{g} = -4\pi G\rho, \tag{5.3}$$

or, equivalently,

$$\nabla^2 \phi = 4\pi G\rho. \tag{5.4}$$

This example of Poisson's equation is the field equation of Newtonian gravity.

Finally, we can write the equation of motion as

$$\frac{d^2\vec{r}}{dt^2} = \vec{g} = -\nabla\phi. \tag{5.5}$$

However, while Newtonian gravity is very successful to a high degree of precision, it is ultimately incomplete since it is an action at a distance law[1].

5.2 Gravitational Redshift

By Einstein's famous expression $E = mc^2$, a photon has an effective mass $m_{ph} = E_{em}/c^2$, where $E_{em} = h\nu_{em}$ is the energy of the photon at the point of emission r. Since anything with mass (or in this case, effective mass) must do work against gravity, then at the point of observation, the observed wavelength, given from

$$E_{obs} = h\nu_{obs} = \frac{hc}{\lambda_{obs}}$$

must be longer. The work done is

$$\begin{aligned} W &= -\int_R^{+\infty} \vec{F} \cdot d\vec{r} \\ &= \frac{GMm_{ph}}{r} = \frac{GME_{em}}{rc^2}, \end{aligned}$$

so

[1]Newton himself was bothered by this, and left it to future generations to work out the problem.

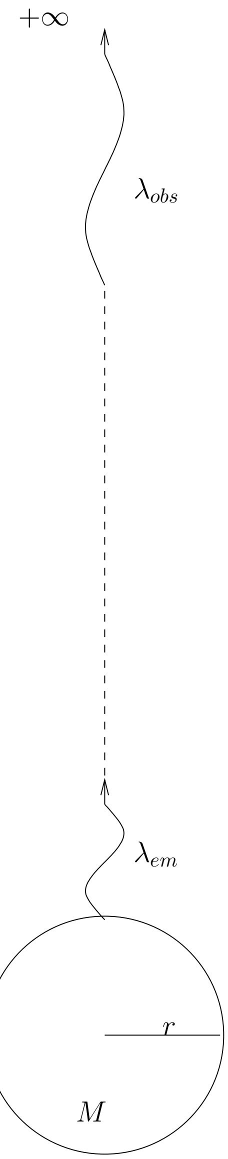


Figure 5.2: In gravitational redshift, a photon's wavelength is stretched.

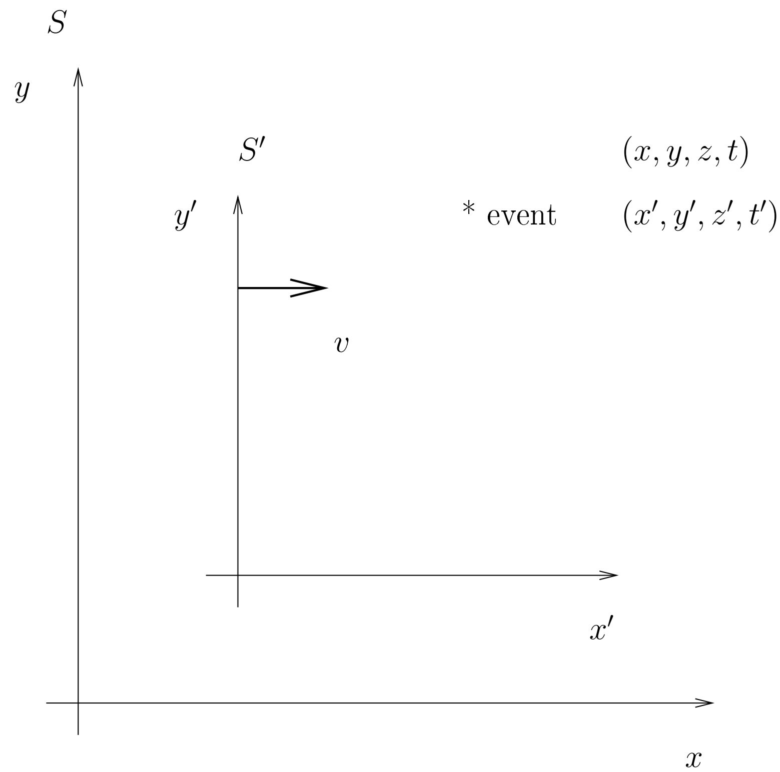


Figure 5.3: An event occurs in two different inertial frames of reference, S and S'. S' is moving with respect to S at constant velocity v.

$$E_{obs} = E_{em}\left(1 - \frac{GM}{rc^2}\right).$$

In terms of frequency[2],

$$\nu_{obs} = \nu_{em}\left(1 - \frac{GM}{rc^2}\right). \tag{5.6}$$

5.3 Special Relativity

Consider any given event in different inertial[3] systems, one of which is moving with constant veloctiy v in the x direction. That is, each intertial frame has its own time and space coordinates (Figure 5.3). In order to connect the coordinates in each frame, we use the *Lorentz transformation*[4]:

[2]Sometimes, this relation may be written as $\nu_{obs} = \nu_{em}(1 + \phi/c^2)$.
[3]"Intertial" meaning unaccelerated.
[4]Here, the primed coordinates represent the coordinates in the frame that is moving with respect to "us," the unprimed coordinates.

$$\begin{aligned}
y' &= y \\
z' &= z \\
x' &= \frac{x - vt}{\sqrt{1 - v^2/c^2}} \\
t' &= \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}.
\end{aligned}$$

Often we consider two events A and B and figure out the difference in coordinates

$$\begin{aligned}
\Delta x &= x_B - x_A \to dx \\
\Delta t &= t_B - t_A \to dt
\end{aligned}$$

In such a case, the above Lorentz transformation works just as well for "delta" quantities.

Example: Time Dilation

Say we have a clock whose rest frame S' is moving with velocity v with respect to us. Then $\Delta t' = \tau$ is the *proper time*. Also, because S' is the clock's rest frame, $\Delta x' = \Delta y' = \Delta z' = 0$. Then, connecting to our reference frame,

$$\begin{aligned}
\Delta t' &= \frac{\Delta t - (v/c^2)\Delta x}{\sqrt{1 - v^2/c^2}} \\
&= \frac{\Delta t(1 - v/c^2)}{\sqrt{1 - v^2/c^2}} \\
&= \Delta t \sqrt{1 - \frac{v^2}{c^2}}.
\end{aligned}$$

Thus the origin of the sloppy expression, "moving clocks run slower."

5.4 The Spacetime Interval

Time and space are dependent on the frame of reference. So then what is "real" in special relativity? The answer came from one of Einstein's old math professors, Hermann Minkowski. In 1908, Minkowski noticed that

$$(\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2 - c^2(\Delta t')^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - c^2(\Delta t)^2. \tag{5.7}$$

He had discovered an invariant quantity under coordinate transormations! Usually we write this as

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2. \tag{5.8}$$

Note that the last three terms add like the "normal" 3 dimensional distance formula. Also, note that there is still a fundamental difference between space and time in that time has the opposite sign associated with it as space.

So what really is this spacetime? It is the 4 dimensional[5], "real" arena for all of physics. That is, it is more "real" than local coordinates are, since these depend on one's motion. Spacetime is the collection of all events (a *manifold*). It is oftentimes useful to draw spacetime diagrams. An example is shown in Figure 5.4.

[5] (t, x, y, z).

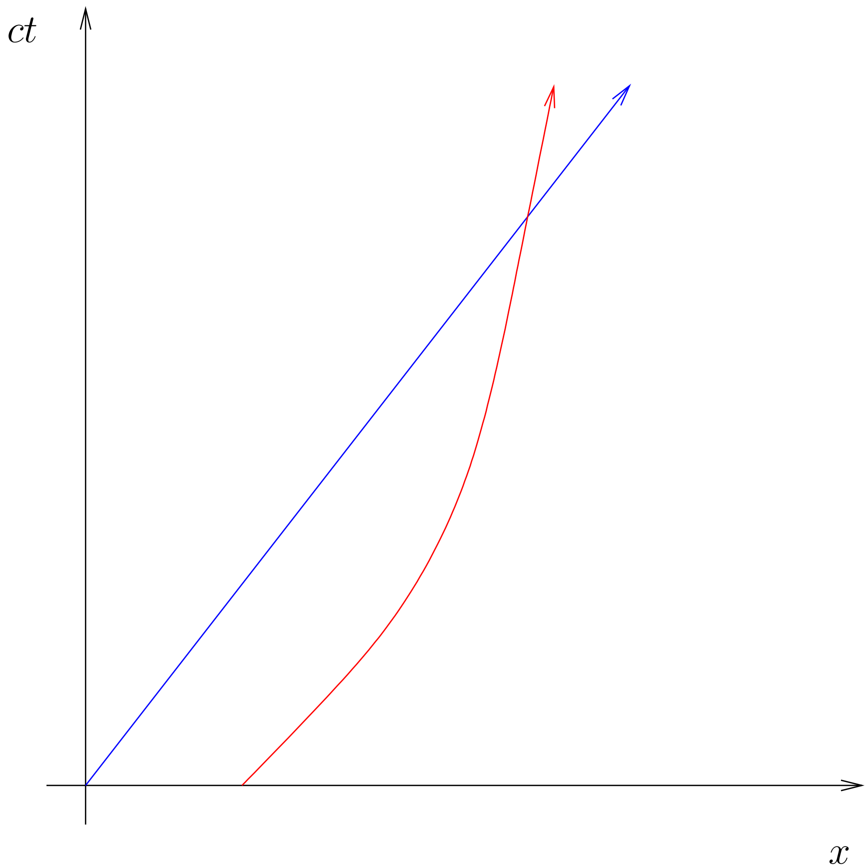


Figure 5.4: A spacetime diagram on the xt plane. Each line represents the "world line" of a particle. Photon world lines (blue) are always straight since they must travel at speed c; *cf.* the world line of a massive particle (red).

There are a few notational simplifications for the spacetime interval. One way to write it is

$$ds^2 = \sum_{\mu,\nu} \eta_{\mu\nu} dx^\mu dx^\nu, \tag{5.9}$$

where μ, ν are integers on the interval $[0, 3]$ and $dx^0 = dt$, $dx^1 = dx$, and so on. The *Minkowski metric*, $\eta_{\mu\nu}$ is

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.10}$$

An even more compact way to express the spacetime interval is to drop the summation sign altogether:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu. \tag{5.11}$$

This notation is known as the *Einstein summation rule* — summation always occurs over repeated indices.

The Minkowski metric demonstrates that the spacetime of special relativity is flat in that the metric coefficients do not depend on location. That is,

$$\frac{d\eta_{\mu\nu}}{d\xi^\lambda} = 0,$$

where ξ is any variable. This is not true of other geometries, for example a spherical one in which case

$$dl^2 = r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2.$$

5.5 Gravity

After formulating special relativity, Einstein set out to find a relativistically correct theory of gravity. He began by using his famous elevator analogy to demonstrate that gravity can (almost) be transformed away.

Imagine an elevator at rest on the surface of the earth (Figure 5.5(a)). If you drop a ball of mass m, it will fall to the ground with an acceleration g. Now compare this to a freely falling elevator (Figure 5.5(b)). Now the ball doesn't move with respect to the walls. In the free fall frame, the laws of physics are the same as if gravity were completely absent. This is known as the *equivalence principle*. From this principle, we can say that gravity is (almost) a fictitious force.

However, there is one manifestation of gravity that *cannot* be transformed away. Consider two balls freely falling towards the earth (Figure 5.6). At first they seem to be falling in parallel, but because they are both attracted to the *center* of the earth, they slowly start to converge! This is an example of a *tidal effect*. These tidal effects are similar to the behavior of straight lines on curved surfaces (e.g., on the surface of the earth, a geodesic is the shortest path between any two points).

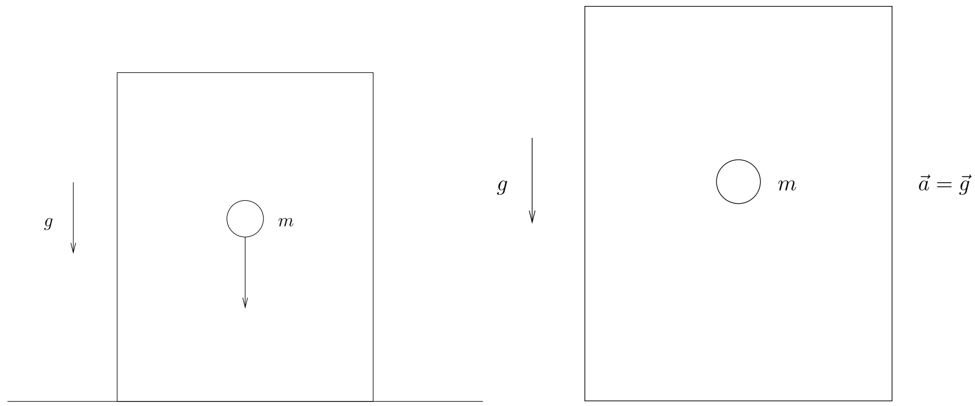
This led Einstein to propose gravity as the curvature of spacetime. We begin to connect gravity with the geometry of spacetime by reformulating Newton's theory in the language of spacetime. Recall from the gravitational redshift that

$$\nu_{\text{obs}} = \nu_{\text{em}} \left(1 - \frac{GM}{rc^2}\right).$$

This also tells us something about time. Consider a "light clock." Since

$$\lambda_{\text{obs}} = \frac{c}{\nu_{\text{obs}}} = c\Delta t_{\text{obs}},$$

we have



(a) In an elevator at rest on the surface of the earth, dropping a ball results in its falling with acceleration g.

(b) In an elevator freely falling with acceleration g, letting go of the ball results in it remaining stationary with respect to the walls — just like astronauts in the space shuttle.

Figure 5.5: Einstein's elevators.

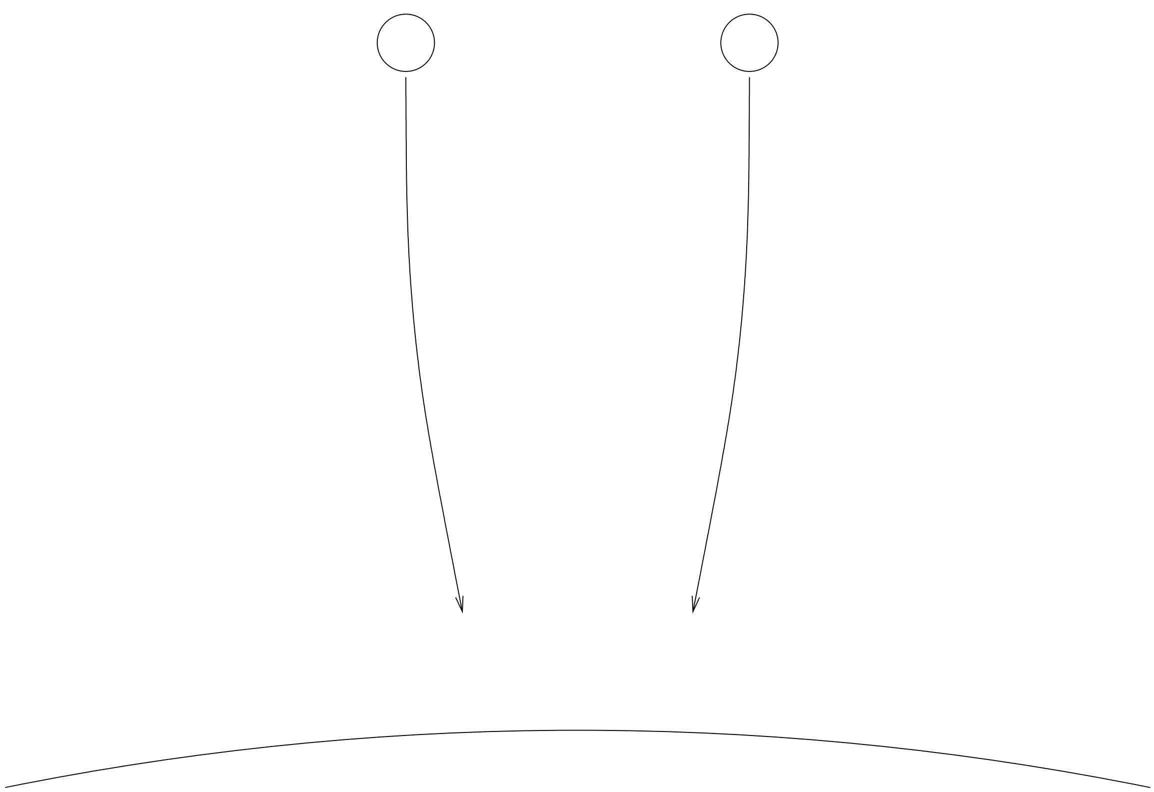


Figure 5.6: As two balls fall towards the center of the earth, they slowly begin to converge.

$$\frac{1}{\Delta t_{\mathrm{obs}}} = \frac{1}{\Delta t_{\mathrm{em}}}\left(1 - \frac{GM}{rc^2}\right).$$

Or,

$$\Delta t_{\mathrm{em}} = \Delta t_{\mathrm{obs}}\left(1 - \frac{GM}{rc^2}\right). \tag{5.12}$$

Recall now that the proper time τ is time measured by the clock at rest, i.e., $\Delta t_{\mathrm{em}} = \Delta\tau$. Now say $\Delta t_{\mathrm{obs}} = \Delta t =$ "coordinate time," the time we measure at $+\infty$. Then

$$\Delta\tau = \left(1 - \frac{GM}{rc^2}\right)\Delta t. \tag{5.13}$$

Now connecting this with the spacetime interval, we note that for the clock in the rest frame, $dx = dy = dz = 0$, so

$$\begin{aligned} ds^2 &= -c^2 d\tau^2 \\ &= -c^2\left(1 - \frac{GM}{rc^2}\right)^2 \Delta t^2. \end{aligned}$$

For gravity in the solar system, the fields are weak. That is, $GM/(rc^2) \ll 1$. Then we can use Taylor expansion and find

$$\left(1 - \frac{GM}{rc^2}\right)^2 \approx 1 - \frac{2GM}{rc^2}$$

(this turns out also to be correct for strong fields, although this is not an obvious fact by any means). Thus we arrive at the spacetime interval from Newtonian theory:

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2 dt^2 + dx^2 + dy^2 + dz^2. \tag{5.14}$$

With $dt = 0$, we have the normal Euclidean space.

Recall now that the Newtonian gravitational potential is $\phi = -GM/r$. Then using this, we rewrite the metric coefficients as $(1 + 2\phi/c^2)$. In general,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu,$$

where $g_{\mu\nu}$ is "the metric" (or "the metric tensor"). So in the GR metric for weak fields,

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2 dt + \left(1 + \frac{2GM}{rc^2}\right)(dx^2 + dy^2 + dz^2).$$

In full GR, both time and space are curved.

Trying to prove Einstein wrong, Sir Arthur Eddington observed the Hyades star cluster during the eclipse of 1919. By Newtonian gravitation, the amount that the light should be bent by (see Figure 5.7) is given by

$$\theta = \frac{2GM}{bc^2}.$$

However, he found that the general relativistic calculation yielded the correct result of

$$\theta = \frac{4GM}{bc^2}. \tag{5.15}$$

This then became the first experimental confirmation of Einstein's results.

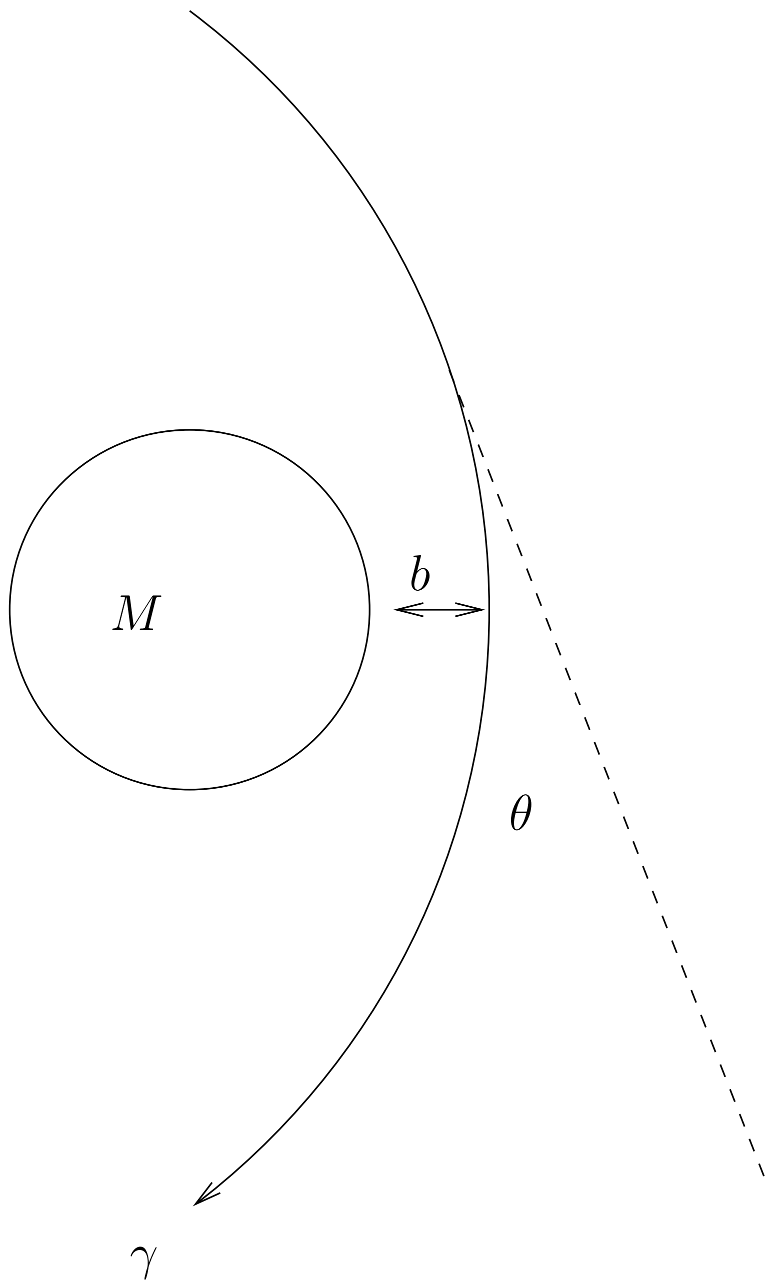


Figure 5.7: A massive object causes light to bend around it (greatly exaggerated). While this effect is predicted by Newtonian mechanics (provided you believe in $E = mc^2$), the resulting angle is half what is (correctly) calculated by general relativity.

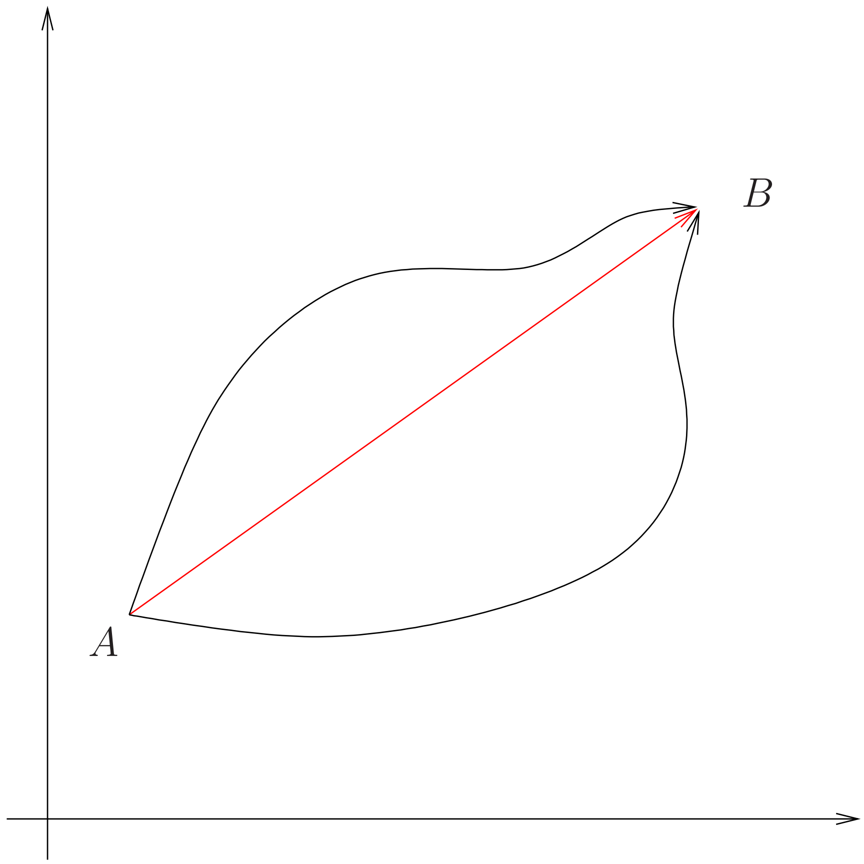


Figure 5.8: Particles tend towards travelling along the shortest path. By using the calculus of variations, we find the shortest path is the one that satisfies $\delta \int_A^B ds = 0$. Above, this occurs along the red path.

5.6 Motion of Particles

From Newton, recall that the equation of motion for a particle in a gravitational field may be written in terms of the gravitational potential as

$$\frac{d^2\vec{r}}{dt^2} = -\nabla\phi.$$

Particles in general travel in the "straightest" possible path (Figure 5.8) — this holds true even in curved spacetime. In such a geometry, the shortest path is along a geodesic.

Consider the motion of a massless photon travelling in the $\hat{x}$ direction. Then $dx = cdt$ and $dy = dz = 0$. Thus the (special relativistic) spacetime interval is

$$ds^2 = -c^2dt^2 + dx^2 = 0. \tag{5.16}$$

We call this the "null geodesic." Photons (and other massless particles) *must* travel along these paths. While it is not transparent, this must also be true in general relativity.

5.7 Relativistic Field Equations

Recall from Newtonian gravity that the field equation is an example of Poisson's equation:

$$\nabla^2\phi = 4\phi G\rho.$$

Evidently, in Newtonian spacetime,

$$g_{00} = -\left(1 - \frac{2GM}{c^2 r}\right) = -\left(1 + \frac{2\phi}{c^2}\right).$$

Then

$$\nabla^2 g_{00} = -\frac{2}{c^2}\nabla^2\phi = -\frac{8\pi G}{c^2}\rho.$$

At this point we must consider the sources of gravity. In Newton's case, the source is simply the mass density ρ. However, in Einstein's case, it is the energy[6] density ρc^2. Thus we also need to include pressure P as a source, since it is also in units of energy/volume. We can organize all of the sources in the form of a matrix:

$$T_{\mu\nu} = \begin{pmatrix} \rho c^2 & & & \\ & P & & \\ & & P & \\ & & & P \end{pmatrix}$$

(usually the elements not filled in are zero). Then the total strength of gravity is from the *effective density*

$$\rho_{\text{eff}} = \frac{1}{c^2}(T_{00} + T_{11} + T_{22} + T_{33}) = \rho + \frac{3P}{c^2}. \tag{5.17}$$

(This is simply the trace of the matrix above).

But now imagine a star with a large mass, and hence large gravity. Then the pressure that is combating gravity itself creates more gravity. To some degree, this can be stable, but eventually, this "problem" can lead to black holes.

With $T_{00} = \rho c^2$, we have

$$\nabla^2 g_{00} = -\frac{8\pi G}{c^4}T_{00}, \tag{5.18}$$

or, more generally,

$$G_{\mu\nu} = -\frac{8\pi G}{c^4}T_{\mu\nu}, \tag{5.19}$$

where

$$G_{\mu\nu} = f(g_{\mu\nu}, \text{first derivatives of } g_{\mu\nu}, \text{second derivatives of } g_{\mu\nu}). \tag{5.20}$$

The general form of Equation 5.19 is given the name of *Einstein's field equations.* In full form, they are incredibly complicated. Because the gravitational field itself contains energy, the gravitational field increseas itself — general relativity is inherently nonlinear!

5.8 Hydrostatic Equilibrium

Now we wish to rephrase hydrostatic equilibrium in terms of general relativity. Recall that before we had (Equation 2.8)

$$\frac{dP}{dr} = -\rho\frac{GM}{r^2}.$$

[6] One could say it is still the mass, since $E = mc^2$, but this way seems somewhat "safer."

Since we don't really have the mathematical tools to modify this rigorously, we shall instead "relativize" it by analogy. We do so in a three step process, as follows:

1. $\rho \to \rho + P/c^2 = \rho[1 + P/(\rho c^2)]$.
2. $m \to m + 4\pi r^3 P/c^2 = m[1 + 4\pi r^3 P/(mc^2)]$.
3. $r^2 \to r^2[1 - 2Gm/(c^2 r)]$.

Putting these steps together, we get the *Oppenheimer-Volkoff equation*[7] (OV equation):

$$\begin{aligned}\frac{dP}{dr} &= -\rho\left(1 + \frac{P}{\rho c^2}\right)\frac{Gm\left(1 + 4\pi r^3 \frac{P}{mc^2}\right)}{r^2\left(1 - \frac{2GM}{rc^2}\right)} \\ &= -\rho\frac{Gm}{r^2}\left(1 + \frac{P}{\rho c^2}\right)\left(1 + \frac{4\pi r^3 P}{mc^2}\right)\left(1 - \frac{2GM}{rc^2}\right)^{-1}. \qquad (5.21)\end{aligned}$$

Note that we can recover our original formulation of the condition for hydrostatic equilibrium by letting $c \to +\infty$ (i.e., by allowing for an action-at-a-distance law of gravity). We can also recover it if the fields are weak, i.e., $\frac{GM}{rc^2} \ll 1, P \ll \rho c^2$.

The OV equation is vitally important to describe neutron stars, compared with white dwarfs, for which the Newtonian formulation suffices.

[7] The second formulation is sometimes more useful pedagogically because it more transparently demonstrates that the Oppenheimer-Volkoff equation for hydrostatic equilibrium is the "normal" equation for hydrostatic equilibrium times some correction factors.

Chapter 6

Neutron Stars

6.1 Discovery

Unlike white dwarfs (Chapter 4) which were discovered before they were even understood, neutron stars were first postulated by theorists. The first to propose these objects were Baade and Zwicky in 1934, only about two years after the discovery of the neutron by Sir James Chadwick. The idea behind neutrons that is now known to be correct is that they are born when very massive stars ($M \geq 8M_\odot$) die in supernova explosions. Baade and Zwicky were mostly ridiculed by the scientific community until Lev Landau published a paper with the same basic concept in 1938.

Neutron stars were first observed in 1967 by the Cambridge astronomers Jocelyn Bell[1] (the graduate student) and Anthony Hewish (the advisor) by way of a *pulsar*. A pulsar is a rotating neutron star whose brightness changes on a time scale of about once ever 1/30 of a second. A famous example of a pulsar is found in the Crab nebula (Figure 6.1).

At first, many astronomers believed the pulsar to be a special type of white dwarf that was literally pulsing (hence the name). However, this was concluded to be impossible for a number of reasons. First, the time scale of pulsation, which is given by τ_{ff} is calculated to be on the order of 1 s, much longer than the observed time scale of the change in intensity for Bell and Hewish's pulsar. Another proposal was that the pulsed intensity was the result of white dwarf rotation. Yet another simple calculation shows that if this were the case, then centrifugal forces would tear it apart. Shortly the scientiific community was forced to accept the fact that pulsars are neutron stars rather than white dwarfs.

6.2 Basic Properties

The basic properties of neutron stars can be deduced from observing pulsars alone. From pulsar binary systems, we see that the mass of neutron stars is $M_{NS} \sim 2M_\odot$. The radius of neutron stars can be estimated by using centrifugal force arguments. That is, the neutron star must be small enough so that it does not fall apart. The period of rotation for a neutron star is

$$T_{NS} \sim \frac{1}{30} \text{ s.}$$

For a massive object of radius R, the rotational speed at the surface is given by

$$v_{\text{rot}} = \frac{2\pi R}{T_{NS}}.$$

Demanding now that the centrifugal acceleration is less than that due to gravity,

[1]When the Nobel Prize committee chose the winners of the 1974 Nobel prize, Bell was famously left out. This was a very controversial decision, and forced the Nobel committe to eventually reevaluate their method of selecting recipients of the award.

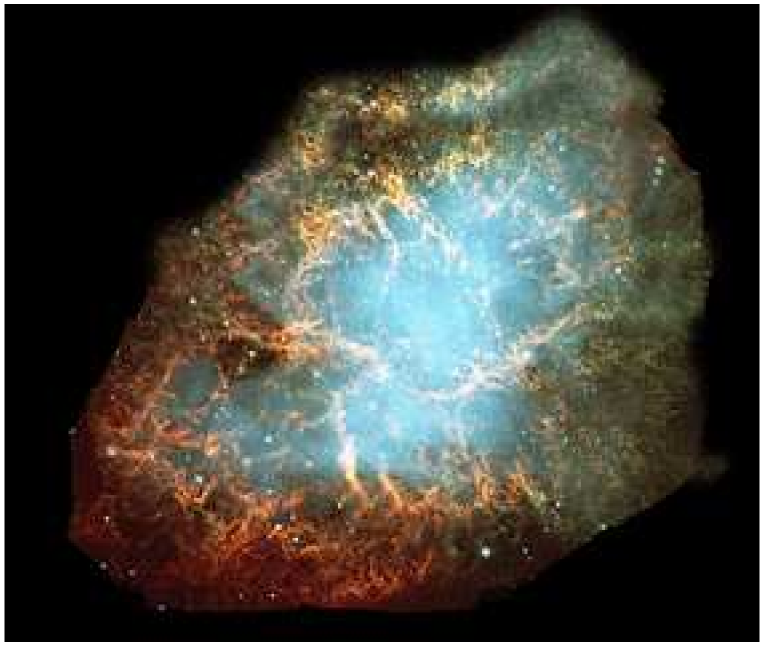


Figure 6.1: The Crab nebula. At the center is a pulsar, the core remnant of the star whose supernova created the nebula. The unique light of the Crab nebula is caused by synchrotron radiation.

Object	**Radius (km)**	**Average Density (g cm^{-3})**
Sun	700,000	1
White Dwarf	7000	10^6
Neutron Star	10	10^{14}

Table 6.1: Approximate orders of magnitude for the radii and average densities of the sun, white dwarfs, and neutron stars.

$$\frac{v_{\rm rot}^2}{R} < \frac{GM}{R^2}.$$

Thus, using this simple argument, we find that

$$R_{NS} \sim \left(\frac{GM_{NS}T_{NS}^2}{4\pi^2}\right)^{1/3} \sim 100 \text{ km}. \tag{6.1}$$

Using more sophisticated arguments, we now know that the radius of neutron stars is more like

$$R_{NS} \sim 10 \text{ km}. \tag{6.2}$$

For comparison, the radii and average densities of the sun, white dwarfs, and neutron stars are given in Table 6.1.

The average density of a neutron star is

$$\langle\rho\rangle_{NS} \sim \frac{M_{NS}}{4\pi/3R_{NS}^3} \sim 10^{14} \text{ cm}^{-3}. \tag{6.3}$$

This is an *extreme* density, as it is in the same range as the nuclear density! In a sense, then, a neutron star can be thought of as a giant atomic nucleus, with one small caveat. While a nucleus such as that of iron is held together by the strong interaction, the neutron star is held together by gravity.

6.3 The Neutronization of Matter

Now we turn to the question of why neutron stars are composed mostly of neutrons. Recall that free neutrons are unstable and undergo beta decay with a half-life of $\tau_{1/2} \sim 10$ minutes:

$$n \rightarrow p + e^- + \bar{\nu}_e$$

In this process, 1.3 MeV is distributed amongst the electron and electron antineutrino, i.e., $(m_n - m_p)c^2 = 1.3$ MeV. The question at hand then becomes the following: Why doesn't this process of beta decay happen in neutron stars?

The answer lies in the fact that neutron decay in neutron stars is energetically disfavored. During the collapse of a massive star en route to becoming a neutron star, the neutrons are bathed in a sea of ultrarelativistic, degenerate electrons. Recall that the Fermi momentum for degenerate electrons is given by

$$p_F = h\left(\frac{3}{8\pi}\right)^{1/3} n_e^{1/3}.$$

From special relativity, recall that for ultrarelativistic particles, $E \approx pc$. Then we define the *Fermi energy* for ultrarelativistic particles as $E_F = p_F c$. In the case of the neutron star, $E_F > 1.3$ MeV, and so it becomes favorable to get rid of electrons via the so-called inverse beta decay (or, perhaps more accurately named, electron capture):

$$p + e^- \rightarrow n + \nu_e$$

Thus, the matter becomes more and more neutron-rich. Hence the origin of the term *neutronization of matter.*

6.4 Pressure Due to Degenerate Neutrons

A quick calculation shows that neutrons become degenerate at $\rho \sim 10^{13}$ g cm^{-3}. Like electrons, neutrons begin degeneracy as nonrelativistic particles, so that the pressure is given by Equation 2.13:

$$P_{NR} = K_{NR} n^{5/3},$$

where recall that

$$K_{NR} = \frac{h^5}{5m}\left(\frac{3}{8\pi}\right)^{2/3}.$$

Here, $m = m_n \approx m_H = 1.67 \times 10^{-24}$ g and $n = n_n$ is the neutron number density. Thus, in a neutron star, the mass density is given by

$$\rho \approx m_H n. \tag{6.4}$$

The neutrons become relativistic (that is, ultrarelativistic and degenerate) for $m_H c^2 \sim pc = p_F c$. This corresponds to a mass density of $\rho \sim 5 \times 10^{14}$ g cm^{-3}.

6.5 Upper Mass Limit

For a neutron star, $R_S/R = 2GM/(c^2 R) \sim 0.5$, i.e., neutron stars are highly relativistic. General relativity is crucial, and we must use the OV equation (Equation 5.21) to describe its mechanical structure. Consider the idealized case of a constant density (i.e., an incompressible fluid) ρ_0 star. Then the mass is a function of r:

$$m = m(r) = \frac{4\pi}{3}\rho_0 r^3.$$

The total mass is

$$M \equiv m(R) = \frac{4\pi}{3}\rho_0 R^3.$$

What follows is an exercise in algebra. Using the OV equation, we can solve for the pressure via separation of variables. Then using the zero boundary condition (pressure is zero at the boundary of the star), we have

$$\int_{\rho(r)}^{0} \frac{dP}{\left(1+\frac{P}{\rho_0 c^2}\right)\left(1+\frac{3P}{\rho_0 c^2}\right)} = -\frac{4\pi G\rho_0^2}{3}\int_r^R \frac{rdr}{1-\frac{8\pi G\rho_0}{3c^2}r^2}.$$

Remarkably, this can be solved analytically. Looking up both integrals in a table and cancelling $\rho_0 c^2$ from both sides,

$$-\frac{1}{2}\left[\ln\left(\frac{1+\frac{P}{\rho_0 c^2}}{1+\frac{3P}{\rho_0 c^2}}\right)\right]_{P(r)}^{0} = \frac{1}{4}\left[\ln\left(1-\frac{8\pi G\rho_0}{3c^2}r^2\right)\right]_r^R.$$

After a bit of algebra, and recalling that the Schwarzschild radius is $R_S = 2GM/c^2$, we wind up with an expression for the pressure as a function of radius,

$$P(r) = \rho_0 c^2 \frac{1-\sqrt{1-R_S/R}}{3\sqrt{1-R_S/R}-1}. \tag{6.5}$$

Evaluating the central pressure at $r = 0$,

$$P_c \equiv P(0) = \rho_0 c^2 \frac{1-\sqrt{1-R_S/R}}{3\sqrt{1-R_S/R}-1}. \tag{6.6}$$

(Compare this to the expression for the central pressure of a Newtonian star, Equation 2.9). Note that the denominator can be zero; i.e., there can, at least mathematically, be singularities! That is, $P_C \to +\infty$ as $R \to (9/8)R_S$. This is a purely general relativistic effect, and implies that $R > (9/8)R_S$ for neutron stars. Since nature cannot have infinite pressure, we see that

$$\frac{9}{8}R_S = \frac{9}{4}\frac{GM}{c^2}.$$

Using $M = (4\pi/3)\rho_0 R^3$, we get

$$R = \left(\frac{3}{4\pi\rho_0}\right)^{1/3} M^{1/3}.$$

(See Figure 6.2).

Now we can solve for the maximum mass of a neutron star by equating the above inequality. The result is called the *Oppenheimer-Volkoff limit* and is given by

$$M_{OV} = M_{max} = \frac{8}{27}\left(\frac{c^2}{G}\right)^{3/2}\left(\frac{3}{4\pi\rho_0}\right)^{1/2} \sim 5M_\odot. \tag{6.7}$$

This limit to the mass of a neutron star is exactly analagous to the Chandresekhar limit (Equation 4.7) for white dwarfs.

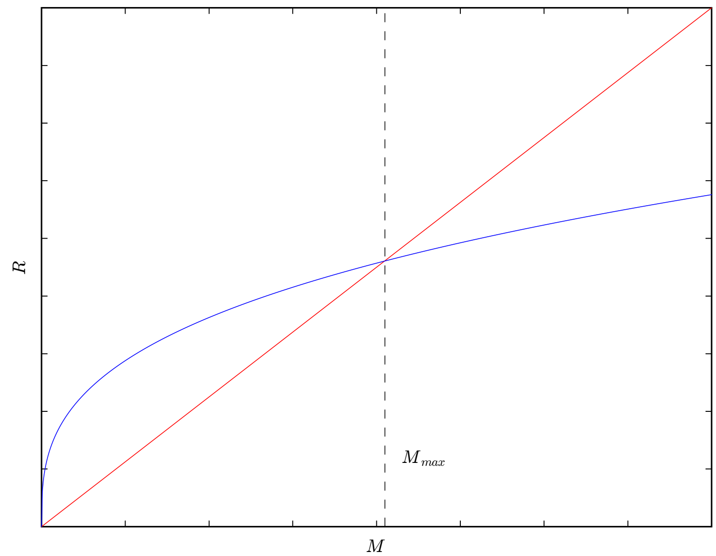


Figure 6.2: The radius versus mass for a neutron star (blue) and the limiting case (red). The region between the curves before they intersect is the allowed region; beyond M_{max} goes against the lack of infinite pressure found in nature.

Revisiting the Upper Mass Limit

Above, we made some "extreme" assumptions. One such assumption involved using an "extreme" equation of state, namely that $\rho = \rho_0 =$ constant. In general, the equation of state for any given matter is $P = K\rho^\gamma$, where γ is the adiabatic index. This implies that

$$\rho = \left(\frac{P}{K}\right)^{1/\gamma},$$

which means that γ must approach $+\infty$ in order that ρ be constant. In general, real equations of state are "softer." That is, the gas *can* be compressed. Even today, the exact equation of state is not well known at such ultrahigh densities.

If it is easier to compress mass than with the above extreme assumption, then the upper mass limit should be lower than we already demonstrated. However, not knowing the exact equation of state means that we are still somewhat unceratin of the precise limit (*cf.* the Chandresekhar limit, which is incredibly precise). Nevertheless, more sophisticated arguments can be made, and today, the Oppenheimer-Volkoff limit is more precisely

$$M_{OV} \sim 1.5 — 3M_\odot. \tag{6.8}$$

Observations

All currently observed neutron star masses cluster around $1.45M_\odot$, which is only slightly above M_{Ch}. There is no known reason why they are all this far below the upper estimate to the limit of $3M_\odot$. Additionally, all observed supermassive stars have $M \gg M_{OV}$ even after their mass losses.

6.6 Structure of Cold, Dead Stars

We construct the equation of state for matter in the ground state for the case where all nuclear fuel is spent (hence "cold, dead matter"). We construct equilibrium stellar models as follows:

- Choose a central density ρ_c.
- Calculate the central pressure P_C from the equation of state.
- Integrate the OV equation dP/dr from the center ($r = 0$) to the surface ($r = R$, which we find from $P(R) = 0$).
- Find $R = R(\rho_c)$, $M = M(\rho_c)$.

(See the Harrison-Wheeler equation of state, figure on notes from 2/23). Any given stellar configuration is stable if $dM/d\rho_c > 0$. Thus, the two stable branches are white dwarfs and neutron stars.

From first principles, we can estimate the radius of *any* cold, dead star. First, we assume that the maximum pressure is under ultrarelativistic conditions. For equilibrium, we have that $E_{\text{pot}} \approx E_{\text{kin}}$, and that $M \sim \rho R^3 \sim m_H n R^3$. We can then estimate the density of particles using the standard technique, namely presuming that the average separation between particles, l, approaches the Compton wavelength, λ_C. Then

$$n^{-1/3} = \frac{h}{mc},$$

where m is m_e for white dwarfs and m_n for neutron stars. Then we see that the mass of a white dwarf or neutron star is given simply by

$$\left(\frac{h}{mc}\right)^{-3} m_H R^3. \tag{6.9}$$

Evaluating the equilibrium condition $E_{\text{pot}} \approx E_{\text{kin}}$ gives us

$$G m_H^2 \left(\frac{h}{mc}\right)^{-3} R^2 \approx mc^2.$$

Here we notice that if we divide both sides by h/c, we get an expression in terms of the gravitational finestructure constant $\alpha_G = G m_H^2/(hc)$. Solving for the radius, we find

$$R \approx \alpha_G^{-1/2} \frac{h}{mc}. \tag{6.10}$$

Putting in the numbers using this relation, we find that $R_{WD} \sim 20,000$ km and $R_{NS} \sim 10$ km, which were the values that we already knew to be correct. But now, if we take the ratio of the radii with Equation 6.10, we find that

$$\frac{R_{WD}}{R_{NS}} \approx \frac{m_H}{m_e} \approx 1836. \tag{6.11}$$

Here we see an example of microphysics determining macrophysics!

6.7 Internal Structure of Neutron Stars

Neutron stars are incredibly interesting due to the exotic physics that they exhibit. For example, neutron stars have both superfluidity and superconductivity. Together, these give rise to strong magnetic fields. Some other, more speculative ideas about neutron star matter is that they contain a large number of hyperons[2],

[2]Baryons with nonzero strangeness. That is, they contain strange quarks

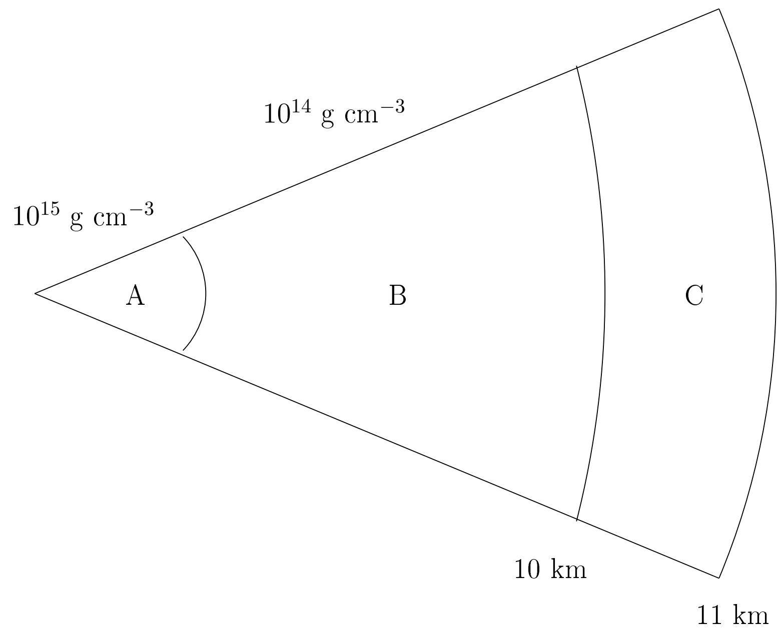


Figure 6.3: **(A)** Solid crust (Fe, n, p, e^-). **(B)** Liquid. This is composed of 99% neutrons, with the neutrons and protons exhibiting superfluidity. **(C)** Unkown. The core could perhaps be solid, or it could be a Bose-Einstein condensate, or it could have undergone hyperonization or.... This is the most unusual and unkown part of the neutron star.

such as the Λ, Σ, and Δ particles. There could even be free quarks (u, d, and s), which could lead to "strange matter" or even "strange stars."

Neutron stars also have what is known as *atypical stratification*. This refers to the layering of different types of matter, as illustrated in Figure 6.3. To sum up, the internal structure of neutron stars gives physicists a probe into matter at ultraextreme conditions.

6.8 Pulsars

Pulsars (first discussed in Section 6.1) have rotational energy $E_{\rm rot} = (1/2)I\omega^2$, where I is the moment of inertia

$$I = \int r^2 dm, \tag{6.12}$$

and $\omega = 2\pi/P$ is the angular frequency of rotation (P is the period — we don't use T in order not to confuse it with temperature, or even kinetic energy). For a solid sphere of constant density, $I = (2/5)MR^2$. Then if we choose $M \sim M_\odot$ and $R \sim 10$ km, we have for a neutron star a moment of inertia of $I \sim 10^{45}$ g cm^2.

Over time, we observe that pulsars slow down; e.g., the Crab pulsar has $P = 33$ s, and so $\omega = 190$ s^{-1} with $dP/dt \approx$ (1ms)/(90years). This is an *incredibly* precise measurement, and is in fact one of the most precise measurements possible in astronomy. Now, the change in angular frequency over time is

$$\frac{d\omega}{dt} = -\frac{2\pi}{P^2}\frac{dP}{dt} \approx -2 \times 10^{-9} \text{ s}^{-2}.$$

The change in rotational energy is then

$$\dot{E}_{\rm rot} = I\omega\frac{d\omega}{dt} \approx -4 \times 10^{38} \text{ erg s}^{-1}.$$

Comparing this to the luminosity of the sun, $L_\odot = 4 \times 10^{33}$ erg s^{-1}, we see that this is more than ten thousand times more energy output than the sun!

We can estimate the age of a given pulsar, τ_{pulsar} using $P \approx 2\dot{P}\tau_{\text{pulsar}}$. Then, the *spin-down age* of a pulsar is

$$\tau_{\text{pulsar}} = \frac{P}{2\dot{P}}. \tag{6.13}$$

For the Crab pulsar, we find $\tau_{\text{Crab}} \sim 1500$ years. This is a fairly reasonable estimate given all the approximations we made since Chinese records indicate[3] the supernova that created the Crab nebula occurred in 1054, so we got the age correct to within a factor of 2.

The big question that still remains is, How is the rotational energy transformed into radiation? That is, what is the mechanism of radiation? In general, electromagnetic radiation is due to accelerated charges. The *Larmor formula* says,

$$\frac{dE}{dt} = \frac{2}{3c^2} q^2 |\vec{a}|^2, \tag{6.14}$$

where $\vec{a}$ is the acceleration. Consider the electric dipole consisting of charges $+q$ and $-q$ separated by $\vec{r}$. Then the electric dipole moment is $\vec{d} = q\vec{r}$ so that $\dot{E} = 2/(3c^2)|\ddot{d}|^2$.

Analagously, we can also have a magnetic dipole. The magnetic dipole moment is

$$\begin{aligned} \vec{\mu} &= \frac{1}{c}(\text{current}) \times (\text{area}) \times \hat{n} \\ &= \frac{1}{c} i\pi R^2 \hat{n}, \end{aligned} \tag{6.15}$$

where R is the radius of an Ampèrian loop of current i. Ampère's law states that

$$2\pi RB = \frac{4\pi}{c} i,$$

so the magnitude of the magnetic dipole is $\mu \approx R^3 B$. Magnetic dipole radiation can then be described by the equation

$$\frac{dE}{dt} = \frac{2}{3c^2} |\ddot{\mu}|^2. \tag{6.16}$$

Consider a sinusoidal variation in the magnetic dipole, namely

$$\mu = \mu_0 \sin(\omega t).$$

Then $\ddot{\mu} = -\omega^2 \mu$ and so

$$\frac{dE}{dt} \approx \frac{2}{3c^2} \omega^4 R^4 B^2. \tag{6.17}$$

Using the same numbers as before, we find that in a neutron star, $B \sim 10^{12}$ G $= 10^8$ T. Compared to the 1 G of the earth's magnetic field (or even the 100 G from the sun), this is huge! There even exit[4] that there exist *magnetars* — pulsars with $B \sim 10^{15}$ G.

To explain where such a strong B field comes from, we turn back to the collapse of the star. Here, magnetohydrodynamics (MHD) dictates that the magnetic flux, $\phi = BA \sim BR^2$, is roughly conserved in

[3]Remarkably, no Europeans recorded this event. The supernova in 1054 would have been incredibly bright — so bright that it would have been visible even in the day. As such, it seems odd that not a single person in Europe felt that it was important enough to write down. This, perhaps, is one of the best pieces of evidence as to how behind Europe was in the middle ages in terms of science compared to the Chinese and Arabs, who both recorded the event.

[4]Robert Duncan (a professor at UT) and Christopher Thompson put forward the magnetar concept in 1992. Magnetars are postulated to be connected to gamma ray bursts.

the plasma of the collapsing star. Simply put, the magnetic field lines are frozen into the motion of the gas/fluid. Then, if the star starts with a given magnetic field strength B_i, it will end with a field of strength

$$B_f = B_i \left(\frac{r_i}{r_r} \right)^2 . \tag{6.18}$$

Say $r_i \sim 10^6$ km, $r_f \sim 10$ km and the initial magnetic field is $B_i \sim 100$ G. Then the final field strength is $B_f \sim 10^{10}$ G, which is fairly close to what we calculated above.

Chapter 7

Black Holes

7.1 History

During the Enlightenment of the late 18^{th} century, two physicists independently introduced the concept of "dark stars," stars which were invisible to far away observers. These dark stars were first set forth by Englishman John Michell in 1783, and later by Frenchman Pierre Simon Laplace in 1796 in his book *Le Systeme du Monde*[1].

Dark stars were dependent on two key ideas: Newtonian gravity and the corpuscular theory of light (also due largely to Newton). To find the escape velocity v_{esc} from a body of mass M, the total energy of a particle of mass m,

$$E_{\text{tot}} = E_{\text{pot}} + E_{\text{kin}} = -\frac{GMm}{R} + \frac{1}{2}mv^2$$

is set to zero. Then one finds that

$$v_{\text{esc}} = \sqrt{\frac{2GM}{R}}. \tag{7.1}$$

If the particle in question is a light corpuscle, then the speed v is always equal to the speed of light, c. Substituting this fact into Equation 7.1, we find that there is a critical radius at which light cannot escape:

$$R_S = \frac{2GM}{c^2}. \tag{7.2}$$

By shear luck, this happens to be the Schwarzschild radius.

For a star of radius R_S, a corpuscle of light may leave the surface of the star, but it will inevitably get pulled back and never reach far away observers. So for a planet nearby, the dark star would actually be visible, whereas for a distant observer, it would be invisible. The crucial difference between the Newtonian dark star and the modern day concept of black hole lies here. In the case of dark stars, light *can* travel beyond R_S, whereas in the case of a black hole, *nothing* can escape beyond R_S.

7.2 The Schwarzschild Geometry

We now wish to use general relativity to determine the geometry of spacetime (i.e., we want to determine the gravitational field) *outside* a spherical body of mass M. We further assume that there is no time dependence,

[1] By the third edition, published in 1806, the reference to dark stars was removed. This was because of Thomas Young's double slit experiments which showed light to be wavelike in nature — a crucial fact given that Newtonian dark stars were dependent on the corpuscular theory of light.

i.e., the gravitational field is a so-called *static field*. For convenience, we shall use spherical coordinates. Then immediately we can presume that the spacetime metric should take the form

$$ds^2 = -A(r)c^2dt^2 + B(r)dr^2 + r^2d\Omega^2, \tag{7.3}$$

where $d\Omega = \sin^2(\theta)d\phi^2 + d\theta^2$.

Next, we need to find a solution to Einstein's field equations. Recall that these take the following form:

$$G\left(\frac{\partial^2 g_{\mu\nu}}{\partial x^\lambda \partial x^\chi}; \frac{\partial g_{\mu\nu}}{\partial x^\lambda}; g_{\mu\nu}\right) = -\frac{8\pi G}{c^4}T_{\mu\nu}. \tag{7.4}$$

The left hand side represents the curvature of spacetime, while the right hand side gives the sources of gravity. For the solution outside of the mass (i.e., the *vacuum solution*), $T_{\mu\nu} = 0$, and so

$$A(r) = g_{00} = \left(1 - \frac{2GM}{c^2r}\right) \tag{7.5}$$

$$= g_{11} = \frac{1}{A(r)} \tag{7.6}$$

Then the *Schwarzschild metric* becomes

$$ds^2 = -c^2\left(1 - \frac{2GM}{c^2r}\right)dt^2 + \frac{dr^2}{1 - \frac{2GM}{c^2r}} + r^2d\Omega^2. \tag{7.7}$$

Note that the $M = E/c^2$ is really the effective mass and not the rest mass.

7.2.1 Interpretation of the Radial Coordinate

One must be careful about the meaning of r in Equation 7.7, as it does *not* measure the distance from the center of the massive body since the mass warps spacetime. With $dt = d\phi = d\theta = 0$, we have

$$ds = \frac{dr}{\sqrt{1 - \frac{2GM}{c^2r}}}.$$

Then the physical (proper) length is found by integrating, and is

$$l = \int_{r=0}^{r} ds = \int_{r=0}^{r} \frac{dr'}{\sqrt{1 - \frac{2GM}{c^2r'}}}.$$

Note also that in the case of weak fields ($R_S/r \ll 1$), $ds \approx dr$.

Instead, the radial coordinate is the *circumferential radius*. Consider the physical area on a surface of constant r outside of the massive body (also, $dt = dr = 0$). Then the total surface area is given by

$$A = \int dA = r^2\int_0^\pi \sin^2(\theta)d\theta \int_0^{2\pi} d\phi = 4\pi r^2.$$

This r then is the correc r to be using.

7.2.2 Flow of Time in the Schwarzschild Geometry

Consider two stationary observers (that is, two observers for which $dr = d\theta = d\phi = 0$) — see Figure 7.1. Observer A sits at a location $r \to +\infty$, while observer B is at a given (finite) distance r from the star. Then the proper time τ is given by

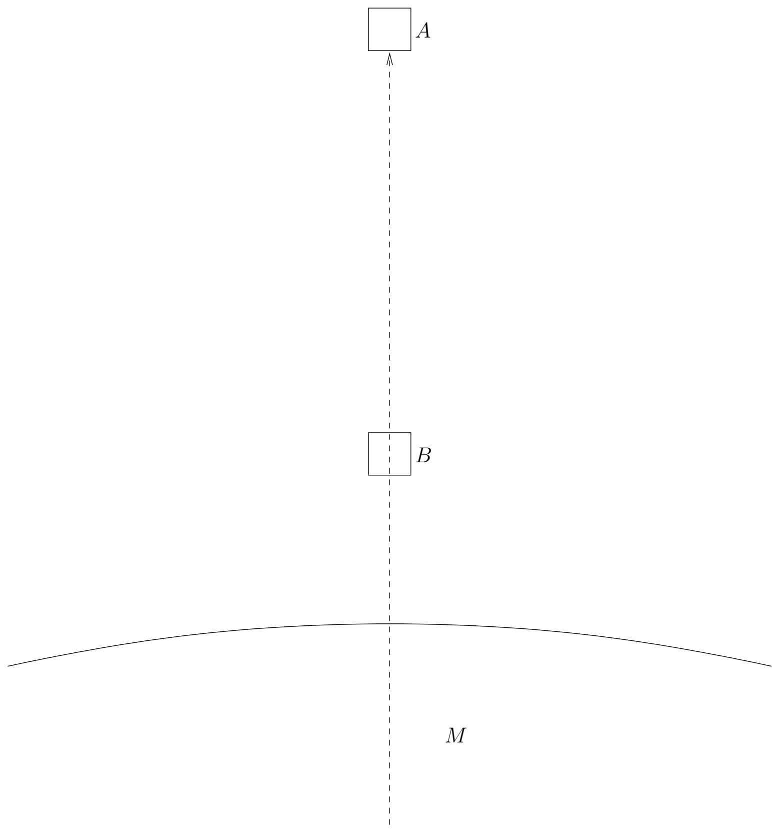


Figure 7.1: Two observers outside a Schwarzschild black hole. A is at a distance $r \to +\infty$ while B is at some finite r. The two observers have very different concepts of time for B.

$$c^2 d\tau^2 = -ds^2.$$

This must be true in *any* geometry, and is allowed because we can choose a small enough free fall frame (recall the equivalence principle from Chapter 5) — locally, we always have the special relativity Minkowski metric, where τ is the time measured by an observer at rest. This observer's clock doesn't change its spatial coordinates, so in this frame, $ds^2 = -c^2 dt^2 \equiv -c^2 d\tau^2$. This is, of course, invariant.

After that slight digression, let's return to our two observers. We wish to compare the flow of time as perceived by B to that of the observer A watching B. Because B has the Schwarzschild geometry, the Schwarzschild metric applies:

$$ds^2 = -c^2 \left(1 - \frac{R_S}{r}\right) dt^2 + \frac{dr}{1 - \frac{R_S}{r}} + r^2 d\Omega^2.$$

For an observer at rest in the Schwarzschild geometry, $dr = d\Omega = 0$, and so ds^2 becomes

$$ds^2 = -c^2 \left(1 - \frac{R_S}{r}\right) dt^2.$$

Let the proper time at A be τ_∞. Since for $r \to +\infty$, $R_S/r \to 0$, we have for A

$$ds^2 \to -c^2 dt^2 \equiv -c^2 d\tau_\infty^2, \tag{7.8}$$

which means that dt^2 *represents the time measured by an observer far away from the Schwarzschild black hole.*

Now we consider the flow of time for the observer at B. If $\tau(r)$ represents the proper time at a distance[2] r, then

$$ds^2 = -c^2\left(1 - \frac{R_S}{r}\right)dt^2 = -c^2 d\tau^2(r),$$

so that

$$-c^2 d\tau^2 = \left(1 - \frac{R_S}{r}\right)c^2 dt^2. \tag{7.9}$$

Solving for $d\tau$, we find that

$$d\tau = \left(1 - \frac{R_S}{r}\right)^{1/2} d\tau_\infty. \tag{7.10}$$

We see that if $r = R_S$, the clock of B "stops!" More precisely, time slows down drastically deep inside of a gravitational potential well. This effect is known as *gravitational time dilation.*

Effect on Light

At B, say a photon is emitted with wavelength λ_{em}. Observer A detects a photon of wavelength $\lambda_{obs} \gg \lambda_{em}$. The emitted frequency is related to a characteristic time, i.e., $\nu_{em} = \Delta\tau_{em}^{-1}$. Say the photon travels "upwards" along the radial direction so that eventually A observes a photon to have $\nu_{obs} = \Delta\tau_{obs}^{-1}$. Then

$$\nu_\infty = \nu\left(1 - \frac{R_S}{r}\right)^{1/2}.$$

This is precisely the formula for gravitational redshift that we found earlier! However, when $r \to R_S$, note that $\nu_\infty = 0$. That is, the photon is redshifted out of existence!

Note that something remarkable happens for photons emmitted at the Schwarzschild radius: $\lambda_\infty \to +\infty$. It matters not what the emitted frequency was[3]. Since the photon is unable to escape the black hole, we see now that black holes must be invisible.

7.2.3 More on the Schwarzschild Metric

Recall that the Schwarzschild metric is valid for *any* spherically symmetric mass *outside* of the matter itself — the object does not need to be a black hole. Then consider the following cases.

$R \gg R_s$

Take, e.g., the sun.. Here, the radius is much larger than the Schwarzschild radius, and so the Schwarzschild geometry is only valid outside of the sun. Because of this, we do not really expect to see some of the extreme redshifting effects as in Section 7.2.2.

$R > R_S$

For a radius R only slightly larger than the Schwarzschild radius, we likely have a neutron star. In this case, the Schwarzschild geometry becomes more useful in explaining spacetime curvature, but it is not always sufficient.

[2]Remember, r is not really the distance. It is actually the radius of curvature. That said, I will continue to be sloppy and often just call it a "distance."

[3]What happened to the energy? Ultimately, conservation of energy is a tricky subject in general relativity, but suffice it to say that it works out in a sense — we just do not have the tools yet to consider it.

$R < R_S$

Here, we are obligated to have a black hole. Thus, $R < R_S$ becomes the operational definition of a black hole, and the Schwarzschild geometry is at its most useful in describing the physics.

7.2.4 Structure at the Critical Radius

In the Schwarzschild metric, we have two *singularities* (or *pathologies*); one is for when $r \to 0$, while the other is for $r \to R_S$. The first case is a "real" singularity — that is, general relativity actualy breaks down at this singularity. The only hope of fully describing it would be to use a quantum theory of gravity.

The second singularity, however, turns out to only be a *coordinate singularity.* That is, it is an artifact of the coordinate system[4] we chose; we can simply transform away this singularity by using a new, more suitable coordinate system.

To study the geometry near the *event horizon* (another term used for the Schwarzschild radius, since this is where not even light can escape), we choose the following time coordinate:

$$\bar{t} = t + \frac{R_S}{c} \ln \left| \frac{r}{R_S} - 1 \right| \tag{7.11}$$

This is known as the *Finkelstein coordinate*, named after its inventor, David Finkelstein. Now we can write $d\bar{t}$ as

$$\begin{aligned} d\bar{t} &= \frac{\partial \bar{t}}{\partial t} dt + \frac{\partial \bar{t}}{\partial r} dr \\ &= dt + \frac{R_S}{c} \frac{1/R_S}{1/R_S - 1} dr. \end{aligned}$$

With this, we can eliminate the old time coordinate and have the new spacetime interval

$$ds^2 = -c^2 \left(1 - \frac{R_S}{r}\right) d\bar{t}^2 + 2c \frac{R_S}{r} d\bar{t} dr + \left(1 + \frac{R_S}{r}\right) dr^2 + r^2 d\Omega^2. \tag{7.12}$$

With the coordinate singularity now removed, we still need to live with the physical singularity at $r = 0$.

Consider now radial light raws (i.e., $d\Omega = 0$); these always move on a path with $ds^2 = 0$ (null geodesics). Before further discussion, we shall introduce the concept of *light cones.* For simplicity, consider first the special relativity case. See Figure 7.2.

For light cones in the Schwarzschild geometry, we find a quadratic equation in $dr/d\bar{t}$ if we divide everything by $d\bar{t}^2$:

$$\left(1 + \frac{R_S}{r}\right) \left(\frac{dr}{d\bar{t}}\right)^2 + 2c \frac{R_S}{r} \left(\frac{dr}{d\bar{t}}\right) - c^2 \left(1 - \frac{R_S}{r}\right) = 0.$$

Solving, we get the roots

$$\begin{aligned} \left(\frac{dr}{d\bar{t}}\right)_1 &= -c; \qquad (7.13) \\ \left(\frac{dr}{d\bar{t}}\right)_2 &= c \frac{1 - R_S/r}{1 + R_S/r}. \qquad (7.14) \end{aligned}$$

[4] An easy to understand example of such singularities is the north pole. Here, there are an infinite number of longitudes, since this is where they converge. However, if we use a different system to describe positions on the earth, this "singularity" would not exist.

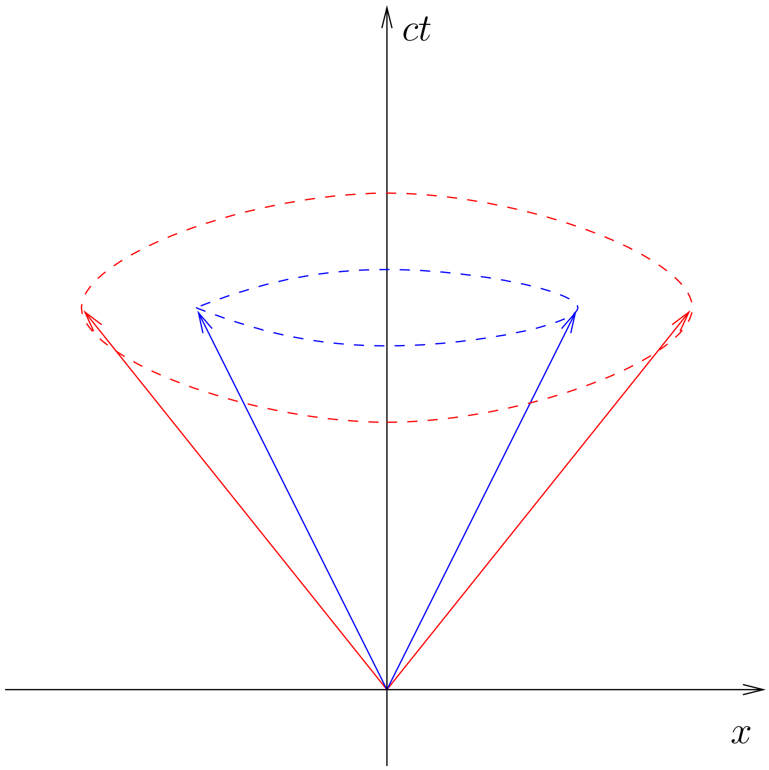


Figure 7.2: Light cones. Photons (red) must travel on null geodesics, where $dx = cdt$ so that $ds^2 = 0$. Massive particles (blue) must always travel inside the light cone of the photon, since $dx/dt = v$ is strictly less than c.

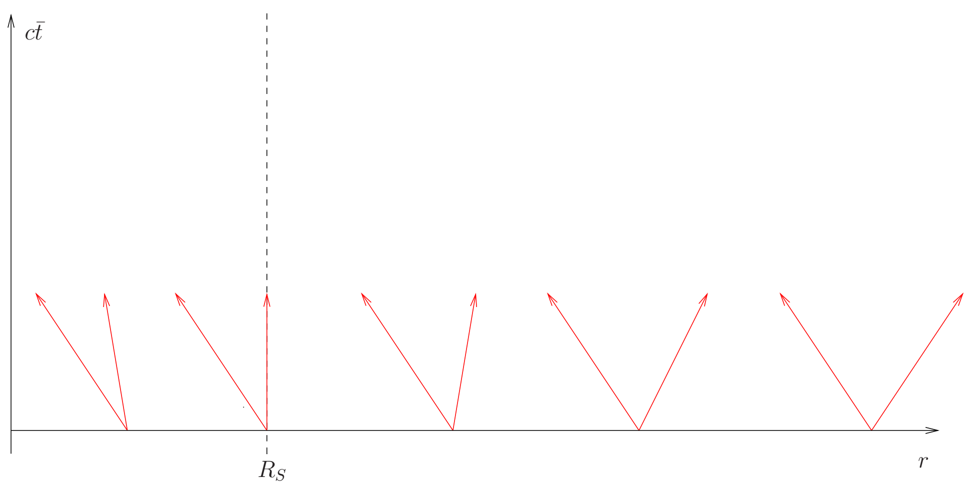


Figure 7.3: Light cones in the Schwarzschild geometry. As $r \to R_S$, the right hand side of the cone gets more and more vertical. At $r = R_S$, the right portion is precisely vertical. At $r < R_S$, both sides of the cone move towards the singularity.

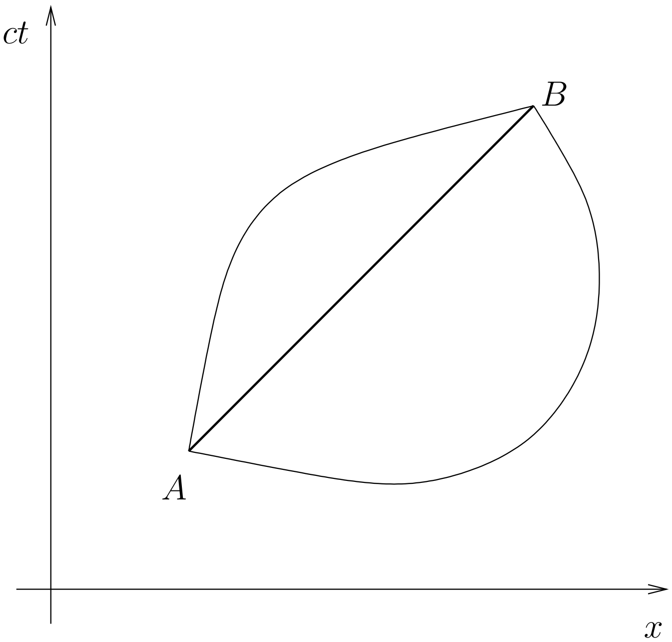


Figure 7.4: In the calculus of variations, we seek out paths such that $\int_A^B ds = \text{extremum}$. Here, such a path would be the one in the middle, the normal "straight line" path.

In Figure 7.3, we see that light cannot escape from the event horizon. This is completely different from the Newtonian concept of the dark star, in which light *does* escape the surface, except eventually gets pulled back. At $r < R_S$, no static equilibrium is possible; anything inside the event horizon is drawn inexorably towards the center.

It is worth noting here the *cosmic censorship conjecture*, proposed by Roger Penrose in 1969. Stated somewhat poetically, "Thou shalt not have naked singularities." That is, every physical singularity must be surrounded by an event horizon. While this is a commonly held belief, to date, there has not been a formal proof, and so it remains a conjecture.

7.2.5 Motion in the Schwarzschild Geometry

Now we return to the concept of the geodesic in a more formal way. Recall that in general relativity, particles move through spacetime along the "straightest" possible path — geodesics. More precisely, between two points A and B, a particle will move along the path for which $\int_A^B ds = \text{extremum}$ (see Figure 7.4). Alternatively, we could formulate this as

$$c\int_A^B d\tau = c\int_A^B \frac{d\tau}{d\tau}d\tau = \text{extremum}.$$

The reason for rewriting this as we do in the second expression will become clear later.

Consider now the Schwarzschild metric in the equatorial plane. That is, in the case with $\theta = \pi/2$ and $d\theta = 0$. Then the metric becomes

$$c^2 d\tau^2 = -ds^2 = c^2\left(1 - \frac{R_S}{r}\right)dt^2 - \frac{dr^2}{1 - R_S/r} - r^2 d\phi^2.$$

If we divide by $d\tau^2$ and define an overdot to mean the "τ derivative," then this becomes

$$c^2 = c^2\left(1-\frac{R_S}{r}\right)\dot{t}^2 - \frac{\dot{r}^2}{1-R_S/r} - r^2\dot{\phi}^2.$$

We then define the Lagrangian[5] as

$$\begin{aligned} L &\equiv L(t,\dot{t},r,\dot{r},\phi,\dot{\phi}) \\ &= \left[c^2\left(1-\frac{R_S}{r}\right)\dot{t}^2 - \frac{\dot{r}^2}{1-R_S/r} - r^2\dot{\phi}^2\right]^{1/2}. \end{aligned} \tag{7.15}$$

Note that $L = c =$ constant. Using the Lagrangian, we can rephrase the extremal condition to be $\int_A^B L d\tau =$ extremum. Now consider two paths that differ in L by ΔL. Then when on the shortest path, we have that

$$\int_A^B (L+\Delta L)\, d\tau - \int_A^B L d\tau = \int_A^B \Delta L d\tau = 0.$$

The difference in the Lagrangian is

$$\begin{aligned} \Delta L &= L(t+\Delta t, \dot{t}+\Delta\dot{t},\ldots) - L(t,\dot{t},\ldots) \\ &= \frac{\partial L}{\partial t}\Delta t + \frac{\partial L}{\partial \dot{t}}\Delta\dot{t} + \ldots \end{aligned}$$

For an extremum, we need

$$\int_A^B \left[\frac{\partial L}{\partial t}\Delta t + \frac{\partial L}{\partial \dot{t}}\Delta\dot{t}\right] d\tau + \int_A^B \left[\frac{\partial L}{\partial r}\Delta r + \frac{\partial L}{\partial \dot{r}}\Delta\dot{r}\right] d\tau = 0.$$

For this to be the case in general, it must also be that each individual integral vanishes independently. We have thus, for the time part,

$$\int_A^B \left[\frac{\partial L}{\partial t}\Delta t + \frac{\partial L}{\partial \dot{t}}\left(\frac{d}{d\tau}\Delta t\right)\right] d\tau = 0.$$

Using integration by parts,

$$\int_A^B \frac{\partial L}{\partial \dot{t}}\frac{d}{dt}(\Delta t) d\tau = \left[\frac{\partial L}{\partial \dot{t}}\Delta t\right]_A^B - \int_A^B \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{t}}\right)\Delta t d\tau.$$

However, the first term goes to zero because all paths converge at A and B (that is, $\Delta t = \Delta r = 0$ at A and B). Thus we have found the Lagrange–Euler equation,

$$\frac{\partial L}{\partial t} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{t}}\right) = 0. \tag{7.16}$$

Of course, we would get similar expressions for the other components of spacetime.

Next, we wish to look at the *constants of motion*. L does not depend on t and ϕ, and so $\partial L/\partial t = \partial L/\partial \phi = 0$, which makes the final equations of motion simpler, because we will have two conservation laws.

First, we have

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{t}}\right) = 0,$$

[5] This is not really a "Lagrangian," in the sense that it is normally used in physics. Instead, it is a "Lagrangian function," since it is a function of each coordinate and its time derivative.

and

$$\frac{\partial L}{\partial \dot{t}} = \frac{c^2 (1 - R_S/r)\, \dot{t}}{L}.$$

Recall that we initially showed that $L = c$, and so we have the first constant of motion to be

$$c^2 \left(1 - \frac{R_S}{r}\right) \dot{t} = \text{ constant.}$$

To interpret the meaning of this, recall from special relativity that the particle energy is given by $E = \gamma mc^2$ where $\gamma = (1 - v^2/c^2)^{-1/2}$ is the Lorentz factor. Using the spacetime interval for motion in one direction, we find the constant of motion to be the *energy per unit rest mass*, i.e., it is a formulation of the conservation of energy:

$$e \equiv \frac{E}{m} = c^2 \left(1 - \frac{R_S}{r}\right) \frac{dt}{d\tau}. \tag{7.17}$$

Note that for $r \gg R_S$, this is the same as the special relativistic energy relation.

Second, we have

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 0,$$

so that

$$\frac{\partial L}{\partial \dot{\phi}} = -\frac{r^2 \dot{\phi}}{L},$$

implying that $r^2 \dot{\phi} = \text{constant}$. We interpret this by recalling angular momentum from Newtonian mechanics. Here, $J = mrv = mr^2 \dot{\phi}$ where $v = \omega r = \dot{\phi} r$. Comparing this to the above, we see that our second constant of motion is *angular momentum per unit rest mass*,

$$j \equiv \frac{J}{m} = r^2 \dot{\phi} = \text{ constant.} \tag{7.18}$$

Note that the general relativistic expression above is in this case no different than the Newtonian expression.

7.3 Stellar Collapse

Oppenheimer and Snyder in 1939 considered massive stars with $M \gg M_{OV}$ — stars far too large to become neutron stars upon collapse. Considering the collapse from the viewpoints of different observers, something strange seems to happen.

Say observer (1) rides on the surface of the collapsing star and observer (2) is at a distance $r \to +\infty$. From (1)'s point of view, nothing strange happens — he falls quickly towards the center as the star collapses. However, from (2)'s point of view, the star collapses until it reaches the Schwarzschild radius — it then stays that way forever!

To begin our analysis, we first note the importance of *Birkhoff's theorem*. It states that the Schwarzschild metric is valid when considering the exterior of *any* spherically symmetric mass distribution, whether it be static or not. That is, we can use the Schwarzschild metric in order to describe a collapsing star.

Consider a radial plunge, i.e., $\dot{\phi} = \dot{\theta} = 0$. Then

$$c^2 = c^2 \left(1 - \frac{R_S}{r}\right) \dot{t}^2 - \frac{\dot{r}^2}{1 - R_S/r}.$$

Exploiting the constants of motion that we derived in Section 7.2.5, we find that

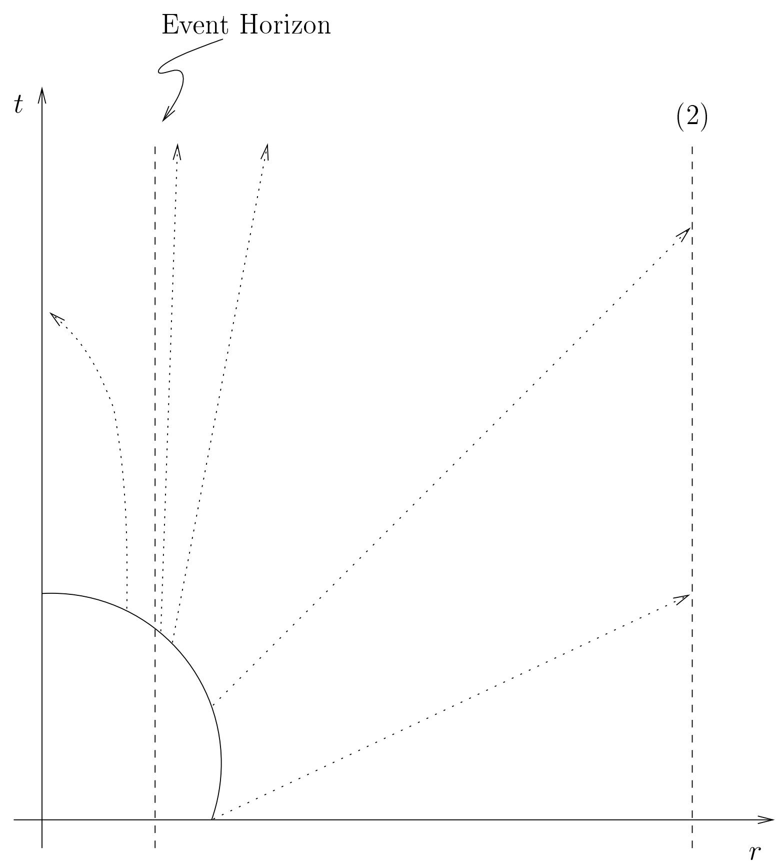


Figure 7.5: A collapsing star from two points of view. Observer (1) rides on the surfcae of the star, while observer (2) is at a far away distance. From (1), photons (dotted lines) are emitted. As the star collapses, the length of time between photon pulses at (2) increases until (1) reaches the event horizon; no more photons are detected at (2), and it seems as if the collapse has frozen.

$$\frac{dr}{d\tau} = -c\left[\frac{e^2}{c^4} - 1 + \frac{R_S}{r}\right]^{1/4}.$$

Say we start the collapse at $r = R_0$ and with an initial velocity of $v_0 = 0$. Then we fix the energy per unit rest mass according to $e^2/c^4 - 1 = -R_S/R_0$. Then we have a separable differential equation for r:

$$\frac{dr}{d\tau} = -c\left(\frac{R_S}{r} - \frac{R_S}{R_0}\right)^{1/2}. \tag{7.19}$$

Finally, putting this into the spacetime interval, we have that

$$\sqrt{\frac{R_0}{R_S}}\frac{dr}{\sqrt{\frac{R_0}{r} - 1}} = -cd\tau.$$

This is cycloid motion! In order to solve this differential equation, we introduce the *cycloid parameter* η, defined as

$$\eta = \arccos\left(2\frac{r}{R_0} - 1\right). \tag{7.20}$$

Then

$$\frac{d\eta}{dr} = \frac{-1}{\sqrt{1 - (2r/R_0 - 1)^2}}\frac{2}{R_0} = -\frac{1}{R_0}\frac{1}{\sqrt{R_0 r - 1}}.$$

Thus, we finally find that

$$\tau(r) = \frac{1}{2}\frac{R_S}{c}\left(\frac{R_0}{R_S}\right)^{3/2}\left[\sin(\eta) + \eta\right]. \tag{7.21}$$

From this, we see that, as we stated earlier, there does not appear to be anything "weird" happening from the perspective of the observer riding on the star's surface. The proper time at $r = 0$ is

$$\tau_0 \equiv \tau(r = 0) = \frac{\pi}{2}\frac{R_S}{c}\left(\frac{R_0}{R_S}\right)^{3/2},$$

so if we have a star of initial radius $R_0 = 5R_S$, then $\tau_0 \approx 17.6 R_S/c$. This also shows us that the characteristic time for black hole collapse is R_S/c,

$$\tau_{BH} = \frac{R_S}{c} = 10^{-5}\ \text{s}\ \left(\frac{M}{M_\odot}\right). \tag{7.22}$$

As for the far observer, we want to find $r = r(t)$. This involves three equations:

$$\begin{aligned} \frac{dr}{d\tau} &= -c\left[\frac{R_S}{r} - \frac{R_S}{R_0}\right]^{1/2} \\ e &= c^2\left(1 - \frac{R_S}{r}\right)\frac{dt}{d\tau} \\ e &= c^2\sqrt{1 - \frac{R_S}{R_0}}. \end{aligned}$$

Combining these, we get

$$\frac{dr}{dt} = -\frac{c}{\sqrt{R_0/R_S - 1}}\sqrt{\frac{R_0}{r} - 1}\left(1 - \frac{R_S}{r}\right). \tag{7.23}$$

Near the event horizon, $r \to R_S$, we get the approximate solution

$$\frac{dr}{dt} \approx -\frac{c}{R_S}(r - R_S).$$

Again we use separation of variables to find the asymptotic behavior to be

$$r(t) \approx R_S + b\exp\left(-\frac{c}{R_S}t\right). \tag{7.24}$$

As we can see, the collapse looks very different to the two observers. Note that for observer (2), the horizon $r = R_S$ is approached asymptotically after an infinite amount of time. Hence, from the viewpoint of a far away observer, the collapse "freezes."

7.4 Orbital Motion

Again we consider motion along the equatorial plane, namely $\theta = \pi/2$ and $d\theta = 0$, leaving the only undetermined variables t, r, and ϕ. We again recall the constants of motion, given by $e = E/m$ and $j = J/m$. With $d\theta = 0$, we have the Schwarzschild metric

$$c^2 = c^2\left(1 - \frac{R_S}{r}\right)\dot{t}^2 - \frac{\dot{r}^2}{1 - R_S/r} - r^2\dot{\phi}^2.$$

Solving for radial motion, we will find that $r = r(\tau)$. Rearranging the above,

$$\left(1 - \frac{R_S}{r}\right)c^2 = c^2\left(1 - \frac{R_S}{r}\right)^2\dot{t}^2 - \dot{r}^2 - \left(1 - \frac{R_S}{r}\right)r^2\dot{\phi}^2,$$

but $c^2(1 - R_S/r) = e^2/c^2$ and $r^2\dot{\phi}^2 = j^2/r^2$, and so

$$\frac{e^2 - c^4}{2c^2} \equiv \epsilon = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}c^2\left[\left(1 - \frac{R_S}{r}\right)\left(1 + \frac{j^2}{c^2r^2}\right) - 1\right].$$

We call the last term the *effective potential*, $V_{eff}(r)$ (see Fig 7.6.

Bound orbits are all in the potential well. A circular orbit occurs at the extremum $dV_{eff}/dr = 0$. That is, we see that

$$\frac{dV_{eff}}{dr} = \frac{GM}{r^2} - \frac{j^2}{r^3} + \frac{3GMj^2}{c^2r^4} = 0.$$

This quadratic equation then has the roots

$$r = \frac{j^2}{2GM}\left[1 \pm \sqrt{1 - 12\left(\frac{GM}{cj}\right)^2}\right]$$

The first one (minus sign) is a maximum in the effective potential (not shown in the figure), and is thus unstable. The second (plus sign) is a minimum, and is where the circular orbits take place. We define the *innermost stable circular orbit* to be that when $r_{min} = r_{max}$, or

$$\frac{cj_{ISCO}}{GM} = \sqrt{12},$$

so

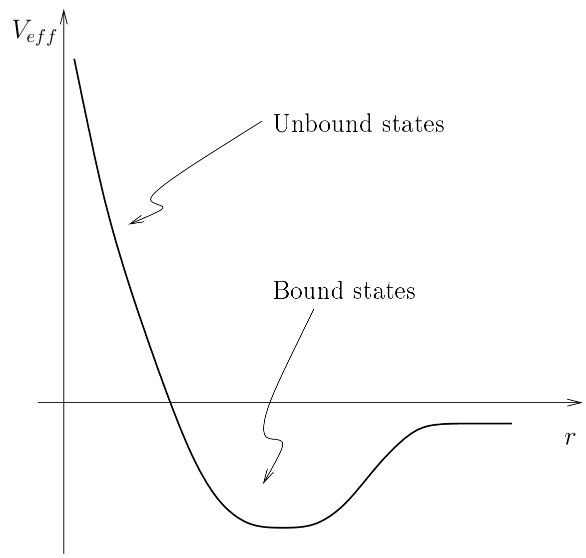


Figure 7.6: The effective potential.

$$r_{ISCO} - \frac{j^2_{ISCO}}{2GM} = \frac{6GM}{c^2} = 3R_S.$$

Thus, once inside a radius of $3R_S$, all matter in an orbit around the black hole will spiral inwards towards the singularity. This is important for determining the energy released by matter accreting in a disc around the black hold.

7.4.1 Black Hole Accretion

If a particle far away is initially at rest and falls towards the black hole, what fraction of its rest energy can be converted into radiation? To answer this question, we must first address the mechanism of radiation. The gravitational potential energy must be converted to heat in a *thermal ring.* If there is no accretion disk, then infalling particles cannot be slowed down by friction and will inevitably collapse into the black hole.

From the virial theorem, $2E_{kin} = -E_{pot}$, so the thermal energy is approximately $-(1/2)E_{pot}$. Ignoring general relativistic effects,

$$E_{\text{thermal}} \approx -\frac{1}{2}\frac{GMm_0}{r} = -\frac{1}{2}\frac{GMm_0}{r_{ISCO}} \approx \frac{1}{12}m_0c^2,$$

or approximately 10% of the rest mass energy. A more precise calculation (i.e., not ignoring GR) gives the value of $0.057m_0c^2$. For comparison, the energy produced in thermonuclear fusion is about $0.007m_0c^2$! So a black hole, by way of its accretion disk, is an incredibly efficient source of energy.

7.5 Generalized Black Holes

All black holes in the universe can be described by three parameters:

1. Mass.
2. Angular momentum.
3. Electric charge (though this is typically unimportant).

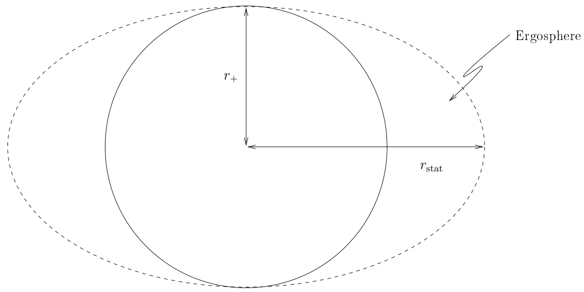


Figure 7.7: A Kerr black hole. The volume contained within r_{stat} is called the ergosphere.

All the other properties of the original star are eventually "forgotten." That is, *black holes have no hair*. This is known as the *no hair theorem*. Essentially, this means that, for example, for a nonrating star that is cube shaped, when it collapses, it will produce a perfectly spherical event horizon surrounding a singularity.

7.6 Rotating Black Holes

We now turn our attention towards the metric for a black hole with given angular momentum $\vec{j}$, mass M, and no electric charge. This was first considered by Roy Kerr in 1963, and so we call this the *Kerr metric*.

The event horizon in a rotating black hole bulges out. It is given by

$$r_+ = \frac{GM}{c^2} + \left[\left(\frac{GM}{c^2} \right)^2 - \left(\frac{j}{Mc} \right)^2 \right]^{1/2}. \tag{7.25}$$

Then, for a nonspinning black hole ($j = 0$), $r_+ = R_S$. Interestingly, for larger j, the event horizon is actually *smaller* than for smaller j. However, we also see that there is a maximum spin, because otherwise there would be a negative square root. The maximal rotation is given by

$$j_{\max} = \frac{GM^2}{c}. \tag{7.26}$$

We interpret this to mean that centrifugal force is exactly balancing gravity.

Close to r_+, spacetime itself is forced to rotate together with the black hole. The *static limit* is given by

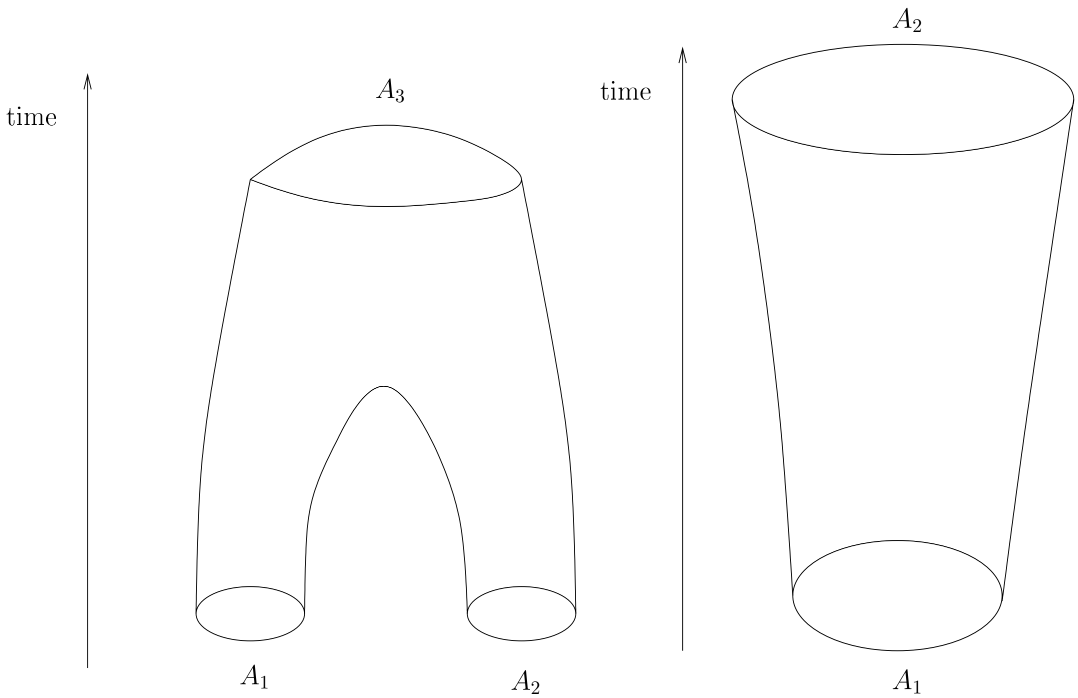
$$r_{\text{stat}} = \frac{GM}{c^2} + \left[\left(\frac{GM}{c^2} \right)^{1/2} - \left(\frac{j}{Mc} \right)^2 \cos^2(\theta) \right]^{1/2}. \tag{7.27}$$

Inside the static limit, there can be no observer that is at rest with respect to the background stars. We call this the *dragging of inertial frames*. We call the volume contained within r_{stat} the *ergosphere*. Because the ergosphere is outside of the event horizon, it is actually possible to extract rotational energy. This is the process that ultimately powers quasars. For a maximally rotating Kerr black hole, the accretion efficiency is approximately $0.42mc^2$.

7.7 Hawking Radiation

7.7.1 Black Hole Area–Increase Theorem

In the 1970s, Stephen Hawking began asking questions about the evolution of black holes over time. His big idea came when considering two black holes (see Figure 7.8(a). Say the surface area of one event horizon is



(a) A black hole merger.

(b) Infalling matter increasing the area of a black hole's event horizon.

Figure 7.8: An illustration of the area–increase theorem.

A_1 and the surface area of the other is A_2. Then if the two merge, the new area A_3 *must* be greater than the sum of the other two areas:

$$A_3 \geq A_1 + A_2. \tag{7.28}$$

Similarly, if we have a single black hole (Figure 7.8(b), infalling matter will cause the area to further increase. That is, for an initial area A_1 and a "final" area A_2,

$$A_2 \geq A_1. \tag{7.29}$$

Using topological methods, Hawking proved that the event horizon area *always* (statistically speaking) goes up with time:

$$\frac{dA_{BH}}{dt} \geq 0. \tag{7.30}$$

7.7.2 Classical Entropy

Recall the second law of thermodynamics: The entropy of an isolated system can never decrease. That is, $dS/dt \geq 0$. This statement is a statistical ones. That is, the laws of physics do not seem to *prevent* the decrease in entropy, it is simply that it effectively *never happens.*

In thermodynamics, entropy was defined by $dS = dQ/T$. Boltzmann's gravestone records his famous statistical mechanical formula for entropy, namely

$$S = k_B \ln(w),$$

where w is the *multiplicity*. That is, w is the number of *microstates* that give rise to a given *macrostate.*

7.7.3 Black Hole Entropy

In 1972, Jacob Bekenstein noticed the similarities between entropy and the black hole area–increase theorem. By analogy, he developed *black hole thermodynamics*. To do so, Bekenstein began by asserting that the entropy of a black hole is proportional to the area of its event horizon:

$$S_{BH} \propto A_{BH}.$$

From a full quantum field theory treatment, he found that the entropy of a black hole could be given by

$$S_{BH} = \frac{1}{4} k_B \left(\frac{A_{BH}}{A_{Pl}} \right), \tag{7.31}$$

where $A_{Pl} = l_{Pl}^2$ is the *Planck area*. So apparently, the black hole horizon *its entropy!*

For a Schwarzschild black hole, $A_{BH} = 4\pi R_S^2$, and the Planck area is

$$A_{Pl} = l_{Pl}^2 = ct_{Pl} = \frac{G\hbar}{c^3}. \tag{7.32}$$

Thus, we find that

$$S_{BH} = k_B \frac{4\pi G}{\hbar c} M^2. \tag{7.33}$$

But then in order for a black hole to be connected with thermodynamics, having an entropy means that a black hole must also have a nonzero temperature. In turn, this implies that a black hole should have a blackbody radiation signature!

Bekenstein noted that

$$T_{BH} dS_{BH} = dQ_{BH} = d(Mc^2) = c^2 dM.$$

Then

$$dS_{BH} = k_B \frac{8\pi G}{\hbar c} M dM,$$

so the effective temperature of a black hole is given by

$$T_{BH} = \frac{\hbar c^3}{8\pi G k_B} M^{-1}.$$

That is,

$$T_{BH} = 10^{-7} \text{ K} \left(\frac{M}{M_\odot} \right).$$

Because this is a nonzero temperature[6], the black hole would *have to radiate*. That means that we need a mechanism for this to occur. Note also that the temperature is inversely proportional to mass.

7.7.4 The Hawking Effect

Consider virtual pair creation near a black hole event horizon (Figure 7.9). In the case where pair production occurs far away from the event horizon, the virtual pair will annihilate quickly in order to conserve energy under the energy–time uncertainty principle.

However, if the pair production occurs in the vicinity of the event horizon, then one of the virtual particles may be drawn into the horizon, leaving the other partner unable to annihilate. In this case, in

[6] Of course, 10^{-7} K is an incredibly *low* temperature (Bose–Einstein condensation occurs at around $10^{-9} K$), so the rate of radiation would be incredibly low.

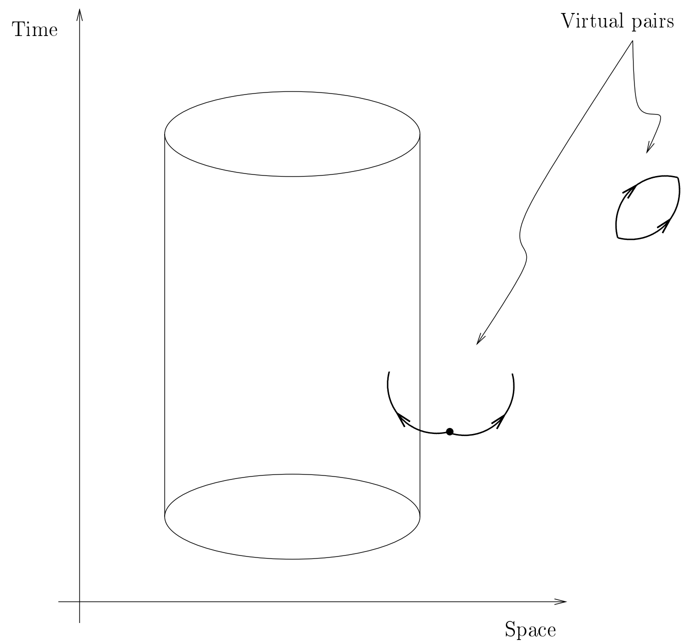


Figure 7.9: In the Hawking effect, virtual pairs are created near a black hole. If a pair is created near the event horizon, the particle nearest the black hole will be drawn in, allowing the other to become a real particle and escape.

order to conserve energy, this partner must become a "real" particle with energy $\epsilon_1 = \epsilon > 0$. Its partner must then have negative energy $\epsilon_2 = -\epsilon < 0$. Through this process, the black hole *loses mass*.

In principle, the pairs produced could be either photons or fermion–antifermion pairs. However, because the photon has no mass, the liklihood that photons are created far exceeds the other possibilities. Because we are considering photons, we also get to use the theory of blackbody (thermal) radiation. Recall that the photon density[7] is given by

$$n_\gamma = \frac{\pi^3 k_B^3}{15\hbar^3 c^3} T^3. \tag{7.34}$$

The mean distance between photons is given by

$$l_{\gamma\gamma} \sim \frac{1}{n_\gamma^{1/3}} \sim \frac{\hbar c}{k_B T}. \tag{7.35}$$

Hawking considered that in order for one of the virtual partners to be near enough the Schwarzschild radius, it must be that $l_{\gamma\gamma} \sim R_S$. From this, we find the temperature of the black hole to be

$$T \approx \frac{\hbar c^3}{G k_B} M^{-1} \approx T_{BH}.$$

[7]There is a simple heuristic argument that can be given to expect the T^3 dependence. We can approximate the "volume occupied" by one photon to be $n_\gamma \sim l_\gamma^{-3}$, where $l_\gamma \sim \lambda$, the photon's wavelength. Then

$$E = h\nu = \frac{hc}{\lambda} = k_B T.$$

Hence, since $\lambda \sim T^{-1}$, we expect that $n_\gamma \sim T^3$.

7.7.5 Black Hole Luminosity

The Stefan–Boltzmann law states in the context of black holes that

$$L_{BH} = 4\pi R_S^2 \sigma_{SB} T_{BH}^4 \tag{7.36}$$

(While this describes the situation fairly well, Hawking derived a slightly modified equation that also takes into account the curvature of spacetime). From this, we find that the luminosity of a black hole is given by

$$L_{BH} \sim 10^{-20} \text{ ergs}^{-1} \left(\frac{M}{M_\odot}\right)^{-2}. \tag{7.37}$$

That is, the lower the mass, the *greater* the luminosity.

Ever so slowly, the black hole with *evaporate*; of course, this happens on a "ridiculous" time scale. This time scale is known as *Hawking time*, τ_{Hawk}, and we find it by setting $d(Mc^2)/dt = -L_{BH}$. This is a trivial differential equation, and so

$$\tau_{\text{Hawk}} \sim 10^{66} \text{ years} \left(\frac{M(t=0)}{M_\odot}\right)^3, \tag{7.38}$$

where $M(t=0)$ is the initial mass. Not unexpectedly, a black hole with relatively low mass will evaporate much quicker than one with a great deal of mass. Yet, even a black hole on the order of a solar mass will take an eternity to evaporate. If we can wait sufficiently long, however, we should observe this happening!

Say we were to observe the complete evaporation of a black hole. We know the age of the universe, the *Hubble time* to be $\tau_H \approx 13.7$ gigayears. Setting this equal to the Hawking time, we get the mass required of a black hole created near the beginning of the universe in order to be just evaporating to be

$$M(t=0) \sim 10^{14} \text{ g}.$$

This is smaller than the sun by several orders of magnitude[8] (recall $M_\odot \approx 2 \times 10^{33}$ g), and so we call this a *mini-black hole.*

In the present universe, black holes are typically on the order of $M_\odot$ because of the Chandresekhar and Oppenheimer–Volkoff limits opposing black hole formation. But a 10^{14} g black hole could have, in principle, been formed in the early universe where the background energy was high enough to squeeze particles together to form small black holes. These primordial mini-black holes may have been possible, then, but if a large number were formed, then we would witness evidence of them all the time. Hence, if mini-black holes do exist, there must only be a small number of them.

7.8 Information Content of Black Holes

If we calculate the entropy of a black hole, we find it to be

$$S_{BH} \sim 10^{77} k_B \left(\frac{M}{M_\odot}\right)^2. \tag{7.39}$$

This is an **immense** amount of entropy! In fact, it is **shockingly huge**. But why is it so large? The answer lies in the no-hair theorem. The various possible shapes of pre-collapsed stars are effectively the microstates to the black hole's macrostate. Then

$$S_{BH} = 10^{77} k_B = k_B \ln(w),$$

so there are $w = \exp(10^{77})$ possible initial states! There is absolutely no way that we can know what the initial state was.

[8] In fact, this is the approximate mass of a mountain on earth

Chapter 8

Observational Evidence for Black Holes

Up until now, we have discussed only the theoretical basis for black holes. In this chapter, we will discuss the experimental evidence that we have for the existence of black holes.

8.1 X-Ray Binaries

8.1.1 Discovery

X-ray binary systems were first discovered by Riccardo Giacconi, et al., in 1962 with X-ray detectors in space[1]. These X-ray probes were simply shot up, took data for a short time, then fell back to earth (X-ray satellite observatories would not arrive till later). Using these probes, incredibly bright X-ray sources were discovered.

8.1.2 Properties of X-Ray Sources

1. Very bright in X-rays.
$$\frac{\Delta E}{\Delta t} \sim 10^{37} \text{ erg s}^{-1},$$
   cf. $L_{\odot} \sim 3.8 \times 10^{33}$ erg s^{-1}.

2. Concentrated towards the Galactic Plane. This implies that the sources are in the Milky Way.

3. Part of a binary system. This is seen in the eclipsing behavior (Figure 8.1). It is presumed that since the X-ray counts completely cease for a time that the X-ray source is small, and it has a large companion.

4. Rapid time variability (Figure 8.2). In this case, $\Delta t_X \sim 1$ ms. From causality, we can constrain the size of the *emission zone*:
$$\Delta R < c\Delta t_X \sim 300 \text{ km}.$$
   This limits the object to being either a neutron star or a black hole[2].

[1] The motivation for Giacconi's research was actually to study the moon in order to determine the viability of landing on its surface.

[2] We know today that both are possible. That is, some X-ray sources are from neutron stars and some are from black holes.

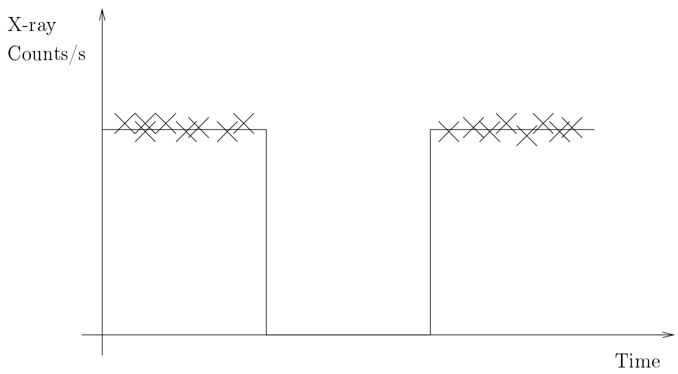


Figure 8.1: X-ray counts per second as a function of time. The regular abscence of counts indicates a binary system.

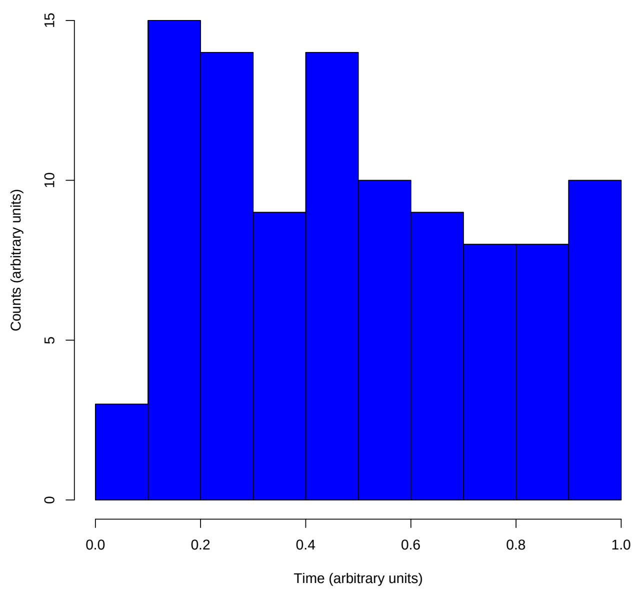


Figure 8.2: Rapid time variability in X-ray sources. This allows one to get an idea of the size of the X-ray source.

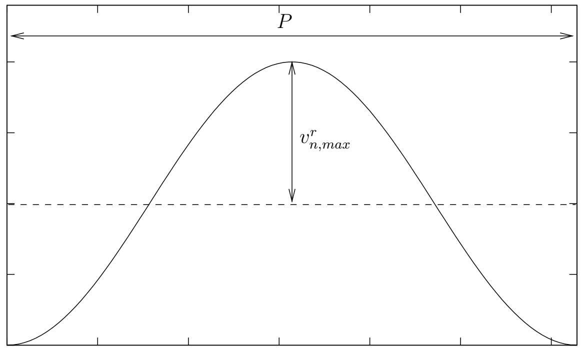


Figure 8.3: The two observables of the X-ray source's companion star.

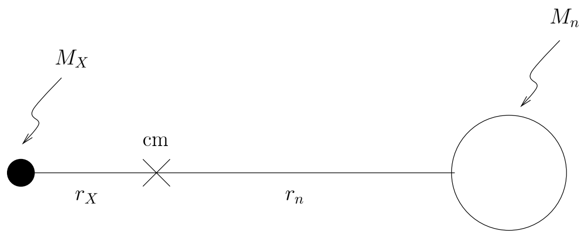


Figure 8.4: The binary system's center of mass frame. We define $r \equiv r_X + r_n$.

8.1.3 Mass Estimate

Using the properties in Section 8.1.2, we can estimate the mass of the X-ray source. If it is greater than M_{OV}, then we will be forced to conclude that it is a black hole.

Because the X-ray source's companion is a normal star, astronomers can use standard optical techniques to measure its radial velocity curve. Then we have two observables: The period P and the maximum radial velocity $v^r_{n,\max}$.

Consider binary orbital motion in the center of mass frame (Figure 8.4). From Archimedes, we know that $M_X r_X = M_n r_n$. Rearranging the terms, we get

$$r_n = r \frac{M_X}{M_n + M_X},$$

where $r \equiv r_X + r_n$. Assuming we can use Newtonian gravitation, the equation of motion in the center of mass frame is given by

$$M_n \frac{v_n^2}{r_n} = \frac{G M_n M_X}{r^2},$$

where $v_n = 2\pi r_n / P$. We then simply get Kepler's third law:

$$\left(\frac{2\pi}{P}\right)^2 r^3 = G(M_X + M_n). \tag{8.1}$$

Now we must connect this with our observables. However, this is not entirely straightforward since we can only observe the line-of-sight component of the X-ray source's velocity. Let i be the *angle of inclination* (see Figure 8.5). For instance, if $i = \pi/2$, then we have an edge-on orbit. Unfortunately, there is no way for us to know i exactly, and so the following estimates will have large error bars associated with them. The maximum velocity is

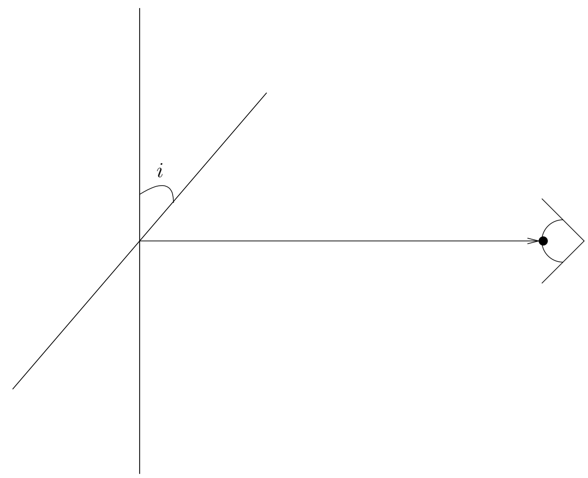


Figure 8.5: The angle of inclination.

$$
\begin{aligned}
(v_n^r)_{\max} &= v_n \sin(i) \\
&= \frac{2\pi r_n}{P} \sin(i) \\
&= \frac{2\pi}{P} r \frac{M_X \sin(i)}{M_n + M_X}.
\end{aligned}
$$

In order to connect this with Kepler's law, it is useful to take the cube:

$$
(v_n^r)^3_{\max} = \frac{2\pi G}{P} r \frac{[M_X \sin(i)]^3}{(M_n + M_X)^2}.
$$

We define the *mass function* to be

$$
\begin{aligned}
f_X(M) &\equiv \frac{[M_X \sin(i)]^3}{(M_n + M_X)^2} \\
&= \frac{P}{2\pi G} (v_n^r)^3_{\max}. \qquad (8.2)
\end{aligned}
$$

Note that everything in the mass function is a directly observable quantity, hence its usefullness.

8.1.4 Cygnus X-1

In 1970, Uhuru, the first real X-ray satellite observatory, discovered an X-ray bright companion to the blue supergiant HDE 226868. From the theory of stellar evolution and structure, we know that the absolute minimum mass of HDE 226868 is $M_n \approx 8.5 M_\odot$. Astronomers can easily trace the radial velocity curve of the large companion to find that $P \approx 5.6$ days, meaning that $(v_n^r)_{\max} \approx 76$ km s^{-1}. Using this information, we find from Equation 8.2 that

$$
f_X(M) \approx 0.25 M_\odot.
$$

Knowing the value of the mass function allows us to get a lower limit estimate of the X-ray bright compact object. Assuming we have an edge-on orbit (i.e., $\sin(i) = 1$),

$$\frac{M_X^3}{(M_X + M_n)^2} \geq 0.25 M_\odot.$$

This may easily be solved numerically to get

$$M_X \geq 3.5 M_\odot. \tag{8.3}$$

This lower limit to the mass of Cygnus X-1 is strong evidence to make it a black hole candidate, since $M_{OV} \geq 2M_\odot$.

8.1.5 X-Ray Power Source

One may wonder why X-rays are emitted rather than radio waves. The answer lies in that the ultimate power source is the accretion of gas encircling the compact object. The *accretion luminosity* is given by

$$L_X \approx L_{\rm acc} \approx \frac{GM}{R}\dot{M}, \tag{8.4}$$

where $R \approx 10$ km if the compact object is a neutron star, and $R \approx r_{ISCO} \approx 3R_S \sim 9$ km$(M/M_\odot)$ if it is a black hole.

There is also a limit on $\dot{M}$, since radiation pressure opposes the infalling matter. Say that the accreting matter has $m = m_H$ As the matter falls inwards, they radiate, producing an X-ray photon flux; gravity must be stronger than the radiation pressure in order for the matter to be able to fall in. Then

$$\frac{GM_X m_H}{r_H} = \frac{\Delta(\text{absorbedphotonmomentum})}{\Delta t} = \frac{\Delta p_x}{\Delta t}.$$

Also,

$$\frac{\Delta E}{\Delta t \Delta A} = \text{flux} = \frac{L_X}{4\pi r^2},$$

where ΔA is an area. For this area, we use the Thomson cross section,

$$\sigma_T \approx 0.66 \times 10^{-24}\ \text{cm}^2. \tag{8.5}$$

Then, since for photons, $E = pc$, we have

$$\frac{\Delta p_x}{\Delta t} = \frac{1}{c}\frac{\Delta E_\gamma}{\Delta t} = \frac{1}{c}\frac{L}{4\pi r^2}\sigma_T.$$

Finally,

$$GM_X m_H = \frac{1}{c}\frac{L_{EDD}}{4\pi}\sigma_T,$$

where L_{EDD} is the *Eddington luminosity*,

$$L_{EDD} = \frac{4\pi G c m_H}{\sigma_T} M \approx 1.38 \times 10^{38} \left(\frac{M}{M_\odot}\right)\ \text{ergs}^{-1}. \tag{8.6}$$

This is quite close to the X-ray luminosity of order 10^{37} erg s^{-1}.

Note that what we have done here is treat two separate particle types as being responsible for only one of the forces. Gravity is much stronger on the protons, and so we ignore gravitational effects. Similarly, radiation pressure is much more powerful on the electrons, and so we ignore the effect on the protons. However, since the two particles are coupled by the electromagnetic interaction, when a force is applied to one particle, the other will follow.

To address the reason for X-ray emission, we look at the blackbody spectrum by estimating the temperature in the vicinity of the compact object. To do so, we make the oversimplifying approximation that the accretion disk behaves like a spherical blackbody. Then we have the simple Stefan-Boltzmann law,

$$L_X \approx \epsilon L_{EDD} \approx 4\pi r^2 \sigma_{SB} T^4.$$

This gives us the rough temperature approximation

$$T \approx 5 \times 10^7 \text{ K} \left(\frac{GM}{c^2 r}\right)^{1/2} \left(\epsilon \frac{M}{M_\odot}\right)^{-1/4}. \tag{8.7}$$

Say the radiative efficiency is $\epsilon \sim 0.5$. Then the temperature in the accretion disk is $T \sim 2 \times 10^7$ K $(M/M_\odot)^{-1/4}$. Counter to intuition, the more massive the black hole, the cooler the accretion disk is!

From this temperature, we can estimate the typical photon energy to be $\bar{E}_\gamma \approx k_B T \sim$ a few keV. Hence, we have X-rays. However, if we did similar calculations for supermassive black holes that are found at the center of galaxies, we would have found that the photons are in the ultraviolet region instead of the X-ray.

Chapter 9

Supernovae

9.1 Phenomenology and History

We classify supernovae as primarily two types:

- Type II. These involve the core collapse of a massive star. The remnant is a neutron star.
- Type Ia. These supernovae result from a thermonuclear explosion of a white dwarf, leaving nothing behind.

In 1987, the nearest supernovae in modern history (SN1987A) occurred in the Large Magellenic Cloud (LMC) about 150,000 light years away. The previous closest supernova dates back to "Kepler's supernova" in 1604.

9.2 Supernovae and Neutrinos

Neutrino detection is one of the tell-tale signs of a core collapse, Type II supernova. Interestingly, this fact was discovered accidentally. Large neutrino detectors such as Kamiokande in Japan were originally built to detect proton decay, which was predicted by some Grand Unified Theories (GUTs). While the proton decay was never discovered, neutrino bursts from supernovae were.

Using neutrino bursts as evidence of far away supernovae, astronomers discovered that the expansion of the universe is accelerating ($\Omega_\Lambda = 0.7$). Supernovae were cruicial to this discovery because Type Ia supernovae are incredibly standardized; they all have approximately the same luminosity since they are all resultant due to white dwarfs which are all around M_{Ch}. Subsequently, we can get a very good estimate to the distance from earth that a Type Ia supernova is.

9.2.1 Energy Resevoir

After all the nuclear fuel of a massive star is exhausted, the remaining core is iron with mass $M_{\rm Fe} \sim 1.5 M_\odot$, temperature $T_C \sim 10^{10}$ K, and density $\rho_{\rm Fe} \sim 10^{10}$ g cm^{-3}. The iron core experiences a rapid loss of energy via neutrino emission (from neutronization of matter) and photodisintigration. The first loss is from

$$p + e^- \to n + \nu_e,$$

while photodisntigration takes the form

$$\gamma + cf.^5 6_2 6 cf.Fe \longrightarrow cf.13cf.^4_2 cf.He + cf.cf.4n. \tag{9.1}$$

The free-fall time of the iron core is

$$\tau_{ff} \sim \frac{1}{\sqrt{G\rho_{\text{Fe}}}} \sim 40 \text{ ms}.$$

Since this is a *much* shorter time scale than He burning, we see that photodisintigration (which creates new He nuclei) is not sufficient to allow the core to continue fusion. Hence, the iron core can no longer support itself against gravity and so it collapses.

We can estimate the energy released during core collapse via the gravitational potential energy. Before the collapse, the gravitational potential energy is

$$E_{\text{pot},b} \approx \frac{GM_{\text{Fe}}^2}{R_{\text{core}}},$$

while after the collapse it becomes

$$E_{\text{pot},a} \approx \frac{GM_{\text{Fe}}^2}{R_{NS}}.$$

The collapse occurs from about 1000 km to about 10 km, so the energy released is

$$\Delta E_{\text{pot}} \approx 5 \times 10^{53} \text{ erg}. \tag{9.2}$$

This is an incredibly large energy! We can compare this to, say, the energy released in the sun's lifetime,

$$E_{\odot} \sim (3 \times 10^{33} \text{ erg}^{-1}) \times (10 \text{ Gyr}) \sim 10^{51} \text{ erg}.$$

Of course, in the case of core collapse, the energy is released in only a few seconds! We could also compare the energy released in the core collapse to the mass energy of the sun, $M_{\odot}c^2 \sim 10^{54}$ erg. That is, the energy released in core collapse is roughly equal to the maximum amount of energy that could be extracted from the sun!

Note that while the above, type II supernova is powered by gravitational energy, type Ia supernovae are powered by thermonuclear energy.

9.2.2 Neutrino Emission

In the iron core, $M_{\text{Fe}} \sim 1.5M_{\odot}$, which means there are on the order of 10^{57} protons, which in turn implies the same number of neutrinos. The energy of each neutrino is about the same as the energy of an electron since the proton and neutron are of the same approximate mass.

We estimate the typical energy of each electron as

$$\bar{\epsilon}_e \sim \epsilon_F \sim p_F c,$$

where ϵ_F and p_F are the Fermi energy and momentum, respectively. Recall that the Fermi momentum is

$$p_F \sim h \left(\frac{3}{8\pi}\right)^{1/3} n_e^{1/3}. \tag{9.3}$$

The density of electrons can be estimated as

$$n_e \sim n_p \sim \frac{\rho_{\text{Fe}}}{m_H} \sim 10^{34} \text{ cm}^{-3},$$

giving the typical energy of an electron in the iron core to be

$$\bar{\epsilon}_e \sim 2 \times 10^{-5} \text{ erg} \sim 10 \text{ MeV}. \tag{9.4}$$

Then we can say the typical neutrino energy is also about 10 MeV.

The *total* energy carried away by neutrino bursts is then

$$E_\nu \sim 10^{57} \times 10 \text{ MeV} = 10^{58} \text{ MeV},$$

or about 10^{52} erg. That is, *a large fraction of the total supernova energy goes into the neutrino burst.*

Appendix A

Useful Astrophysical Constants

Description	**Symbol**	**Value**
Solar mass	$M_\odot$	2×10^{33} g
Solar radius	$R_\odot$	6.9×10^{10} cm
Average solar density	$\langle\rho\rangle_\odot$	1.4 g cm^{-3}
Average white dwarf density	$\langle\rho\rangle_{WD}$	10^6 g cm^{-3}
Average neutron star density	$\langle\rho\rangle_{NS}$	10^{14} g cm^{-3}
Electron mass	m_e	9×10^{-28} g
Hydrogen mass	m_H	1 amu = 1.67×10^{-24} g
Fundamental Unit of Charge	e	4.8×10^{-10} esu
Stefan-Boltzmann Constant	σ_{SB}	5.67×-5 erg s^{-1} cm^{-2} K^{-4}